CIVL7008 Seismic Analysis for Building Structures

Lec-02 Vibration of SDOF System

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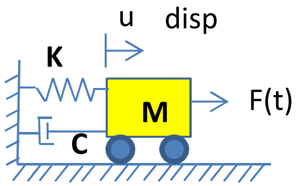
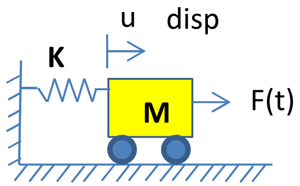
Part 1

Free Vibration of SDOF System

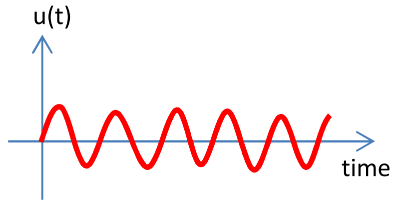
Dynamic System

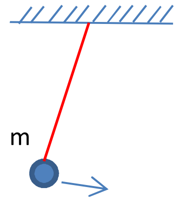


SDOF System



Undamped system

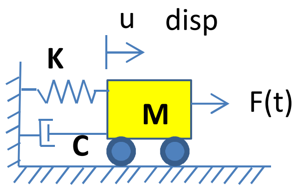




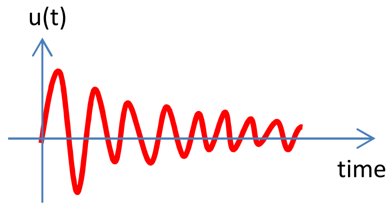
Stiffness, K
Damping, C
Mass, M

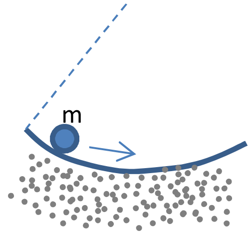
Structure Properties

Dynamic Loading P(t)



Damped System





K: Stiffness Spring Force, Elastic Resistance to displacement

M: Mass, Inertia Force

C: Damping, Energy Loss Mechanism

Equation of Motion

Fp=P(t) = Fi + Fd + Fs

$$m\ddot{u} + c\dot{u} + ku = p(t)$$

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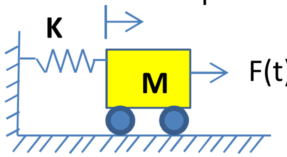
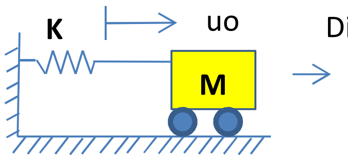
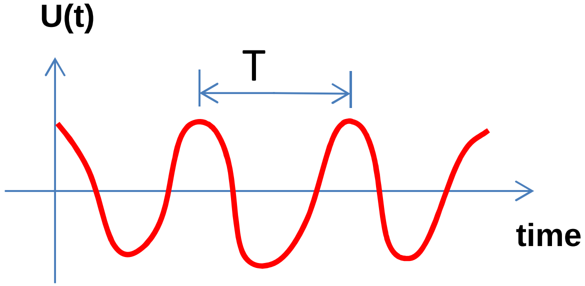
Free Vibration of SDOF (undamped System)



Move to U0
Distance





Free Vibration

Basic Equations

$$m\ddot{u} + ku = 0$$

$$m\ddot{u} + c\dot{u} + ku = p(t)$$

$$C_{cr} = 2m\omega = \frac{2k}{\omega} = 2\sqrt{km}$$

C_{cr} **Critical Damping Constants**

$$\xi = C/C_{cr}$$ **Damping Ratio**

$$\omega = \sqrt{k/m} = 2\pi/T$$ **Circular frequency**

$$T = 2\pi\sqrt{m/k}$$ **Period**

$$f = 1/T$$ **Frequency**

General Solution

$$\begin{cases} u = A_1 cos\omega t + A_2 sin\omega t \\ \dot{u} = -A_1 \omega sin\omega t + A_2 \omega cos\omega t \\ \ddot{u} = -A_1 \omega^2 cos\omega t - A_2 \omega^2 sin\omega t \end{cases}$$

$$u_0 = u(0) = A_1 + 0 = A_1$$

$$\dot{u}_0 = 0 + A_2 \omega = A_2 \omega$$

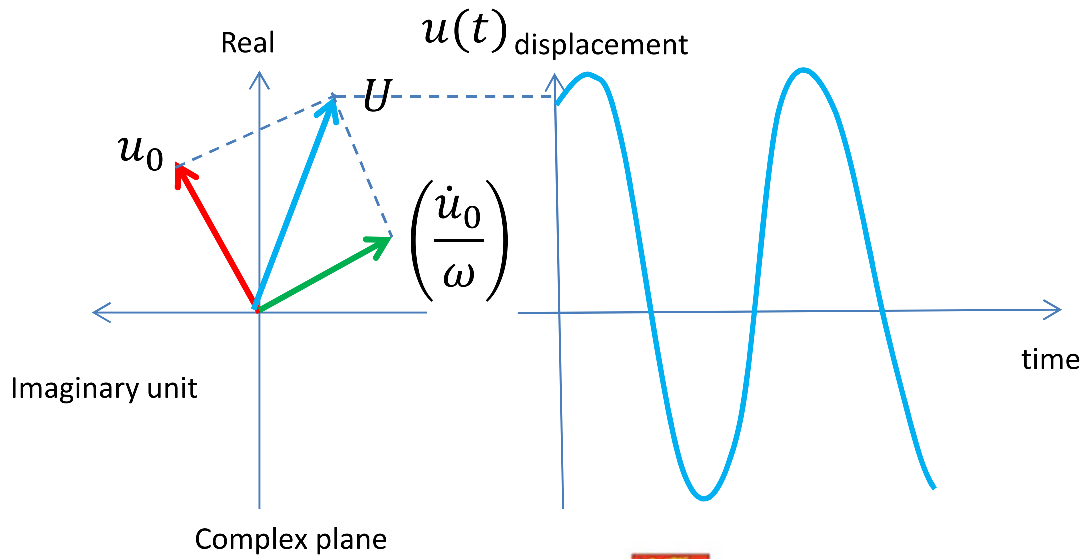
$$A_1 = u_0$$

$$A_2 = \dot{u}_0 / \omega$$

$$u = u_0 cos\omega t + \left(\frac{\dot{u}_0}{\omega}\right) sin\omega t$$

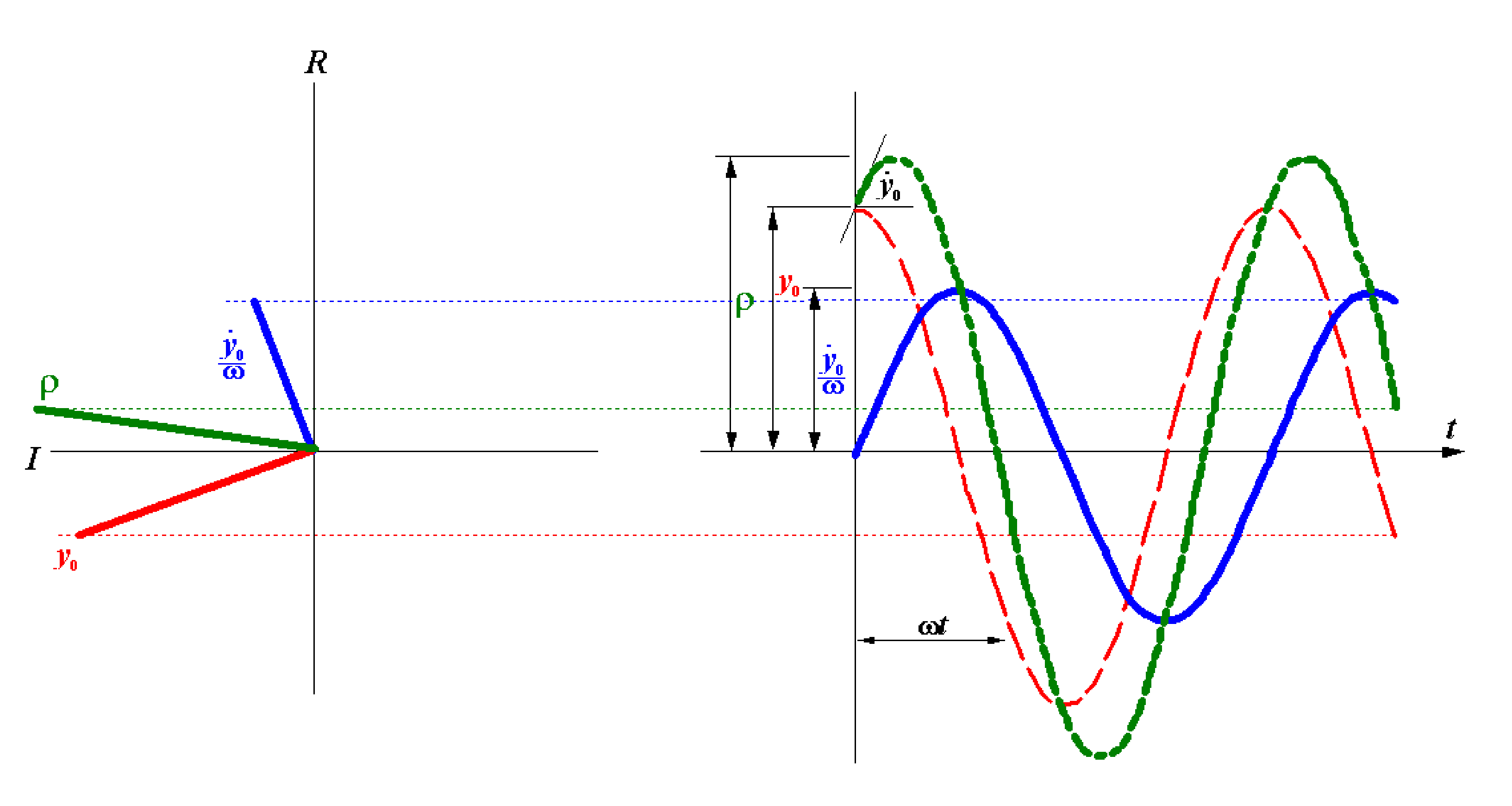
General Solution

$$u(t) = u_0 cos\omega t + \left(\frac{\dot{u}_0}{\omega}\right) sin\omega t$$



Lec-02 Vibration of SDOF System

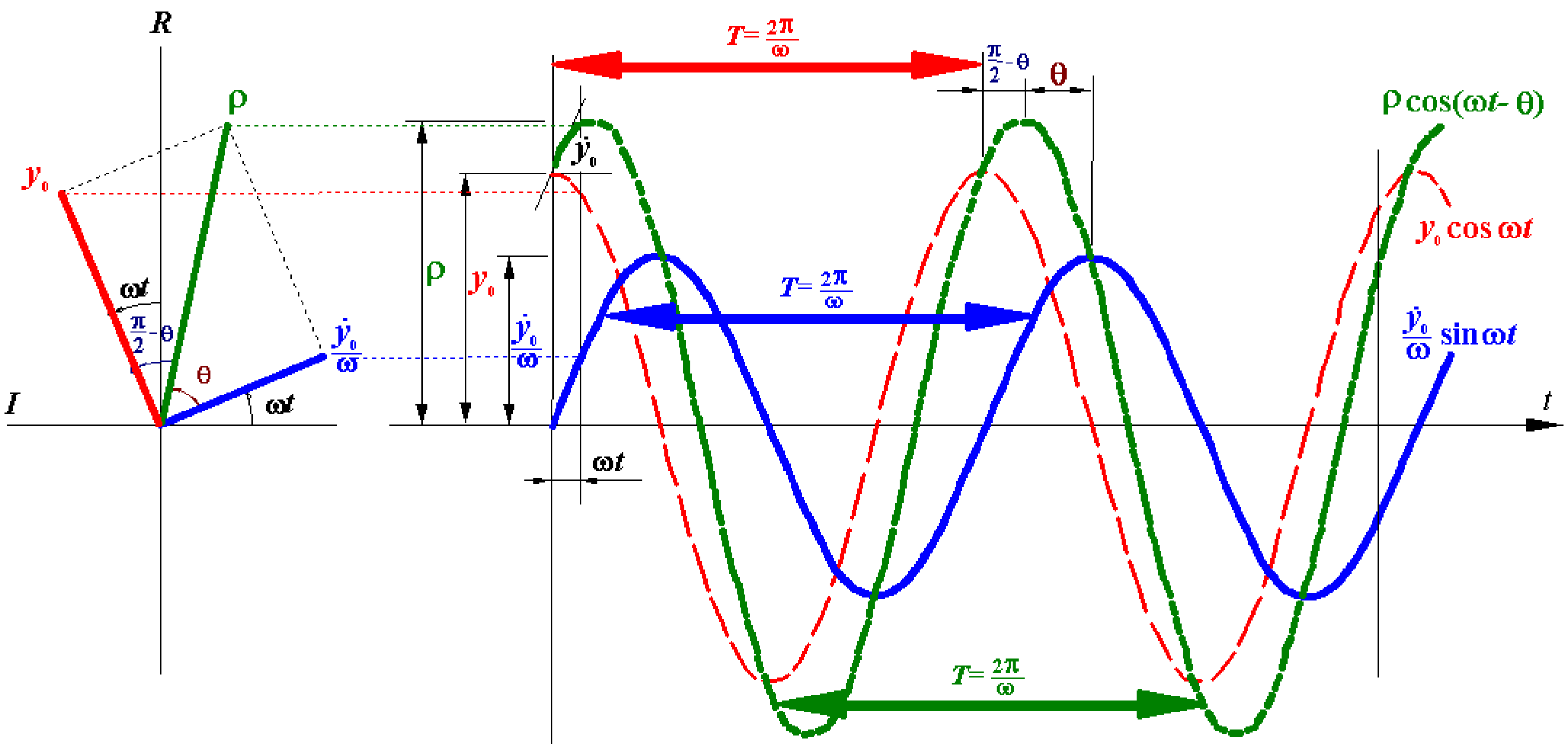
Free Vibration of SDOF (undamped System)



Free Vibration of SDOF (undamped System)

$$v(t) = \frac{\dot{v}_0}{\omega} \sin \omega t + v_0 \cos \omega t$$

$$v(t) = \rho \cos(\omega t - \theta)$$

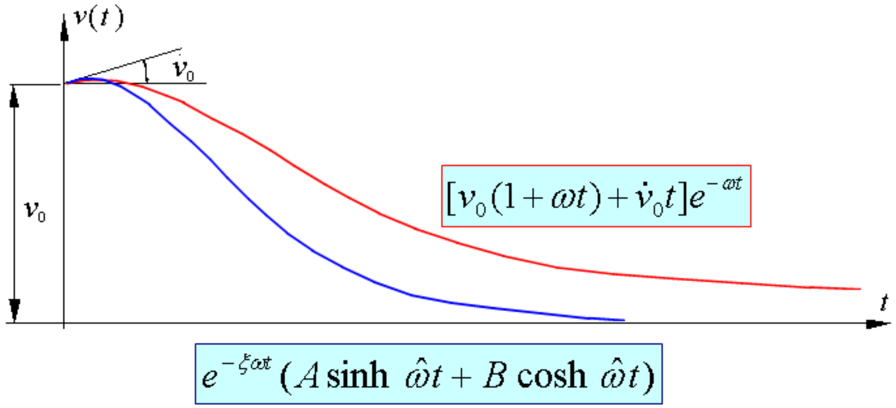


Critical Damping Constant Ccr

$$m\ddot{u} + c\dot{u} + ku = 0$$

$$\rightarrow \ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = 0$$

$$\rightarrow \ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u = 0$$

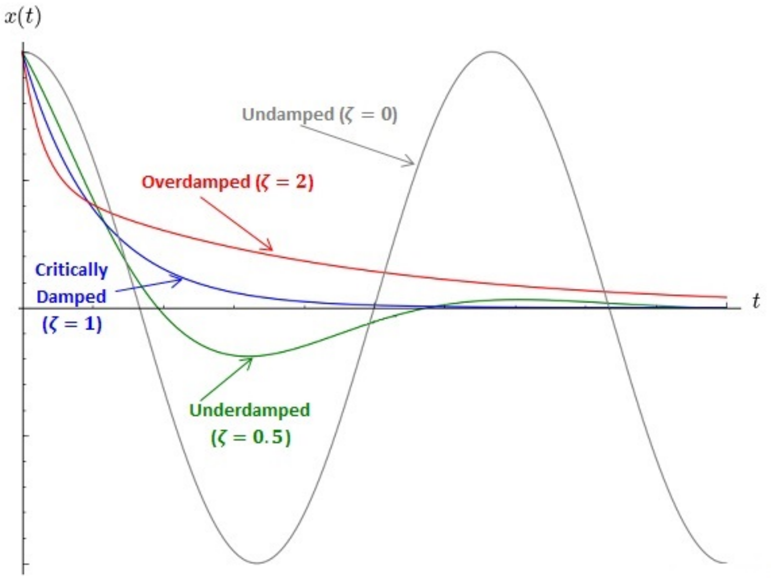


$$Assume\ solution: \quad u = c \cdot exp(s \cdot t)$$

$$\dot{u} = c \cdot exp(s \cdot t) \cdot s$$

$$\ddot{u} = c \cdot exp(s \cdot t) \cdot s^2$$

$$c \cdot exp(s \cdot t) \cdot s^2 + 2\xi\omega_n \cdot c \cdot exp(s \cdot t) \cdot s + \omega_n^2 \cdot c \cdot exp(s \cdot t) = 0$$



$$s^2 + 2\xi\omega_n + \omega_n^2 = 0$$

$$\bar{s}_{1,2} = -\xi\omega_n \pm \sqrt{\xi^2 - 1} \cdot \omega_n = 0$$

$$\xi = 1.0 \quad critial\ damping$$

$$\xi > 1.0 \quad over\ damping$$

$$\xi < 1.0 \quad weak\ damping$$

$$\begin{cases} concrete & \xi = 0.05 \\ steel & \xi = 0.02 \end{cases}$$

Free Vibration(damped system)

$$m\ddot{u} + c\dot{u} + ku = 0$$

$$\omega_d = \omega\sqrt{1-\xi^2}$$

$$u(t) = e^{-\xi\omega t}(A_1 cos\omega_d t + A_2 sin\omega_d t)$$

$$u(0) = u_0 = (A_1 * 1 + A_2 * 0) = A_1$$

$$\dot{u}(0) = \dot{u}_0 = \left(e^{-\xi\omega t}\right)'(A_1 cos\omega_d t + A_2 sin\omega_d t) + \left(e^{-\xi\omega t}\right)(A_1 cos\omega_d t + A_2 sin\omega_d t)'$$
$$= \left(e^{-\xi\omega t}\right)(-\xi\omega)(A_1 cos\omega_d t + A_2 sin\omega_d t) + \left(e^{-\xi\omega t}\right)(-\omega_d A_1 sin\omega_d t + \omega_d A_2 cos\omega_d t)$$
$$= -\xi\omega A_1 + A_2\omega_d$$

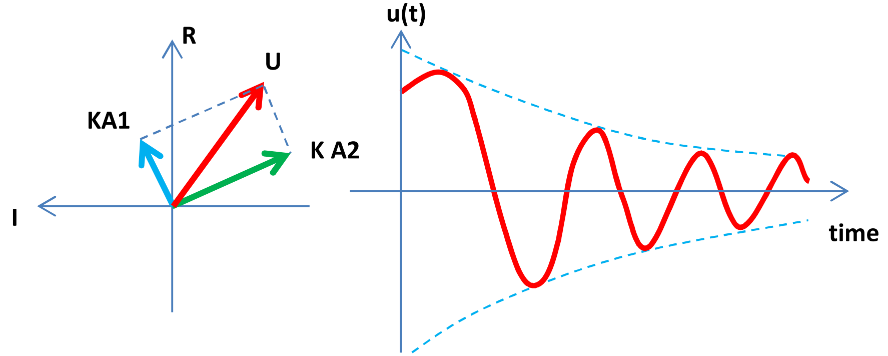
$$A_1 = u_0$$

$$A_2 = (\dot{u}_0 + \xi\omega u_0)/\omega_d$$

$$K = e^{-\xi\omega t}$$

$$u(t) = K(A_1 cos\omega_d t + A_2 sin\omega_d t)$$

Free Vibration(damped system)



$$U = KA_1 + KA_2 = K$$

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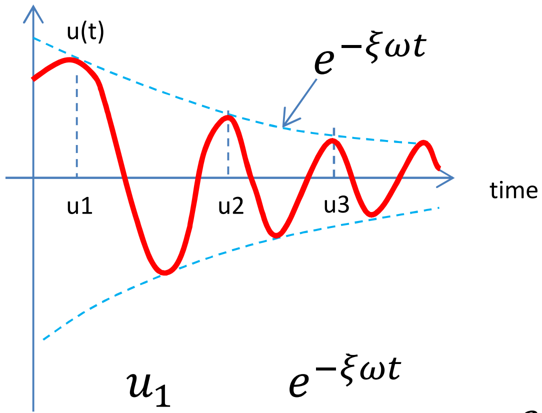
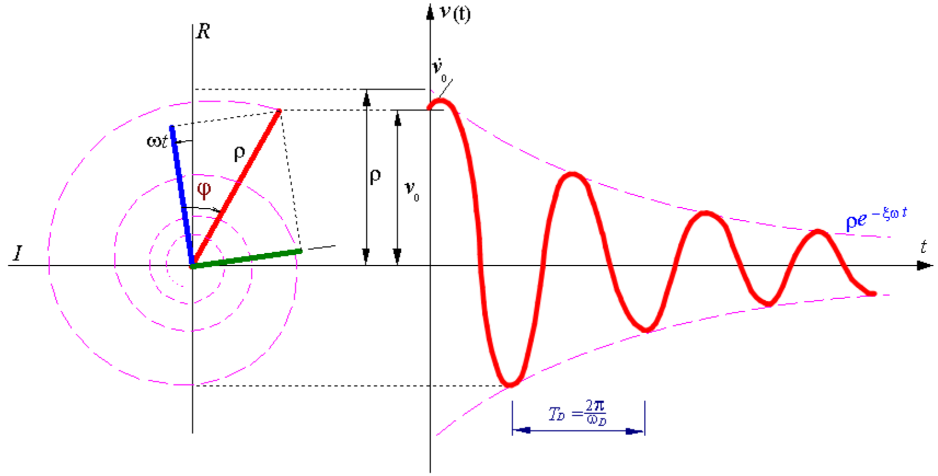


$$v(t) = \rho e^{-\xi\omega t}\cos(\omega_d t - \varphi)$$



$$\frac{u_1}{u_3} = \frac{e^{-\xi\omega t}}{e^{-\xi\omega(t+2T)}} = e^{\xi\omega 2T}$$

$$T = {}^{2\pi}/_{\omega}$$

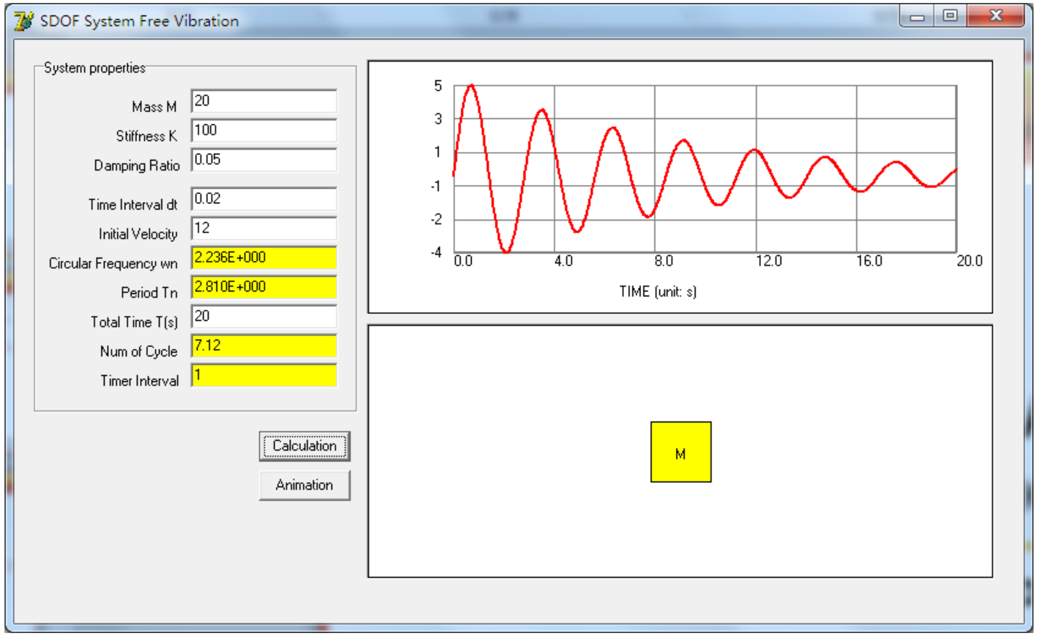
$$\ln({}^{u_1}/_{u_3}) = 2\pi 2\xi$$

$$\xi = {}^{1}/_{2\pi 2} * \ln({}^{u_1}/_{u_3})$$

$$\xi = {}^{1}/_{2\pi n} * \ln({}^{u_1}/_{u_{n+1}})$$

Damping ratio test method

SDOF Simulation
Free Vibration




```
PROCEDURE TFORM1.CALCULATION;
VAR I:INTEGER;
    t:real;
BEGIN
 PI:=3.1415926535;
 M:=STRTOFLOAT(EDIT1.Text);
 K:=STRTOFLOAT(EDIT2.Text);
 DAMP:=STRTOFLOAT(EDIT3.Text);
 DT:=STRTOFLOAT(EDIT4.Text);
 V0:=STRTOFLOAT(EDIT5.Text);
 WN:=SQRT(K/M);
 TN:=2*PI/(WN);
 EDIT6.Text:=FORMAT('%0.4E',[WN]);
 EDIT7.Text:=FORMAT('%0.4E',[TN]);
 ta:=STRTOFLOAT(EDIT8.Text);
 NC:=TA/TN;
 wd:=wn*sqrt(1-damp*damp);
 EDIT9.Text:=FORMAT('%0.2F',[NC]);
 NSTEP:=ROUND(TA/DT);
 C:=2*M*WN*DAMP;
 SETLENGTH(UD1,NSTEP+1);
 //////////////////////////////////////////
 FOR I:=0 TO NSTEP DO
 BEGIN
 t:=i*dt;
 UD1[I]:=(v0/wd)*exp(-damp*wn*t)*sin(Wd*t);
 END;
 //////////////////
 //////////////////
 MAXU:=(v0/wd);
END;
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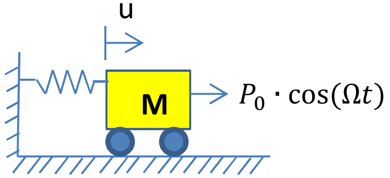
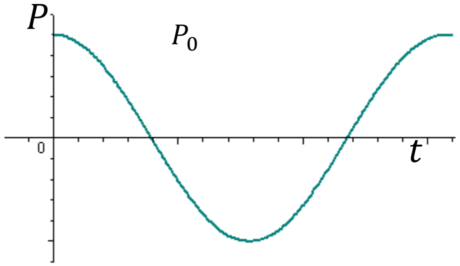
Part 2

Harmonic Vibration of SDOF System

Lec-02 Vibration of SDOF System

Harmonic Vibration (undamped system)





Equation of Motion

$$m\ddot{u} + ku = P_0 \cdot \cos(\Omega t)$$

Particular solution

$$u_p(t) = U \cdot \cos(\Omega t)$$

$$\dot{u}_p(t) = -U \cdot \Omega \cdot \sin(\Omega t)$$

$$\ddot{u}_p(t) = -U \cdot \Omega^2 \cdot \cos(\Omega t)$$

$$\rightarrow m(-U \cdot \Omega^2 \cdot \cos(\Omega t)\ ) + kU \cdot \cos(\Omega t) = P_0 \cdot \cos(\Omega t)$$

$$\rightarrow U = \frac{P_0}{k - m\Omega^2}$$

Static deflection: $U_0 = \frac{P_0}{k}$

Frequency ratio: $r = \frac{\Omega}{\omega_n}$

$$r^2 = \frac{\Omega^2}{\omega_n^2} = \frac{m\Omega^2}{k} \Rightarrow m = \frac{kr^2}{\Omega^2}$$

$$\frac{U}{U_0} = \frac{P_0}{k - m\Omega^2} \cdot \frac{k}{P_0} = \frac{k}{k - m\Omega^2} = \frac{k}{k - kr^2} = \frac{1}{1 - r^2}$$

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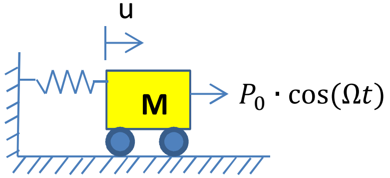
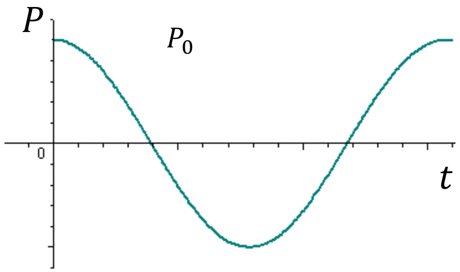
Harmonic Vibration (undamped system)





Frequency-Response Equation

$$H(\Omega) = \frac{U}{U_0} = \frac{1}{1-r^2} \quad \rightarrow r = \frac{\Omega}{\omega_0}$$

$$U_p = \frac{P_0}{k - m\Omega^2} \cdot \cos(\Omega t) = U\cos(\Omega t)$$

$$= H(\Omega) \cdot U_0 \cdot \cos(\Omega t)$$

$$= \frac{1}{1-r^2} \cdot U_0 \cdot \cos(\Omega t)$$

$\frac{1}{1-r^2}$ → Steady Response

Forced Vibration

Solution = Particular + General

Vibration Response = Steady Response + Free Vibration Response

$\Longrightarrow$

$$u(t) = \frac{U_0}{1-r^2} \cdot \cos(\Omega t) + A_1\cos(\omega_n t) + A_2\sin(\omega_n t)$$

$if\ u_0 = u(0) = 0,\ \dot{u}_0 = \dot{u}(0) = 0\ (initial\ condition)$:

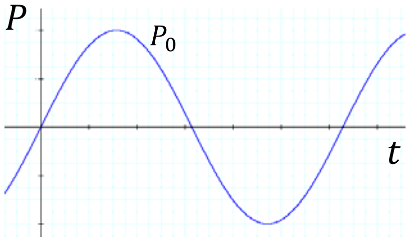
$$\dot{u}(t) = -\frac{U_0\Omega}{1-r^2} \cdot \sin(\Omega t) - A_1\,\omega_n \sin(\omega_n t) + A_2\omega_n\cos(\omega_n t)$$

$$\because \dot{u}(0) = A_2\omega_n\cos(\omega_n t) = 0$$

$$\therefore A_2 = 0$$

$$u(0) = \frac{U_0}{1-r^2} + A_1 = 0$$

Lec-02 Vibration of SDOF System



Equation of Motion

$$m\ddot{u} + ku = P_0 \cdot \sin(\Omega t)$$

Particular solution

$$u_p(t) = U \cdot \sin(\Omega t)$$

$$\dot{u}_p(t) = U \cdot \Omega \cdot \cos(\Omega t)$$

$$\ddot{u}_p(t) = -U \cdot \Omega^2 \cdot \sin(\Omega t)$$

$$\rightarrow m(-U \cdot \Omega^2 \cdot \sin(\Omega t)) + kU \cdot \sin(\Omega t) = P_0 \cdot \sin(\Omega t)$$

$$\rightarrow -mU\Omega^2 + kU = P_0$$

$$\rightarrow U = \frac{P_0}{k - m\Omega^2}$$

Static deflection: $U_0 = \frac{P_0}{k}$

$$r = \frac{\Omega}{\omega_n} \qquad P_0 = kU_0$$

$$U = \frac{kU_0}{k - m\Omega^2} = \frac{U_0}{1 - \frac{m}{k}\Omega^2} = \frac{U_0}{1 - (\frac{\Omega}{\omega_n})^2} = \frac{U_0}{1 - r^2}$$

Frequency-Response Equation

$$H(\Omega) = \frac{U}{U_0} = \frac{1}{1 - r^2}$$

$$U_p = \frac{U_0}{1 - r^2} sin(\Omega t)$$

$$u(t) = \frac{U_0}{1 - r^2} \cdot \sin(\Omega t) + A_1 \sin(\omega_n t) + A_2 \cos(\omega_n t)$$

$$\dot{u}(t) = \frac{U_0 \Omega}{1 - r^2} \cdot \cos(\Omega t) + A_1\, \omega_n \cos(\omega_n t) - A_2 \omega_n \sin(\omega_n t)$$

Lec-02 Vibration of SDOF System

$if\ u_0 = u(0) = 0,\ \dot{u}_0 = \dot{u}(0) = 0\ (initial\ condition)$:

$$\because u(0) = A_2\omega_n \cos(\omega_n t) = 0$$

$$\therefore A_2 = 0$$

$$\because \dot{u}(0) = \frac{U_0\Omega}{1-r^2} + A_1\omega_n = 0$$

$$\therefore A_1 = -\frac{U_0 r}{1-r^2}$$

$$u(t) = \frac{U_0}{1-r^2}\cdot\sin(\Omega t) + A_1\sin(\omega_n t) + A_2\cos(\omega_n t)$$

$$\rightarrow u(t) = \frac{U_0}{1-r^2}\cdot\sin(\Omega t) - \frac{U_0 r}{1-r^2}\sin(\omega_n t)$$

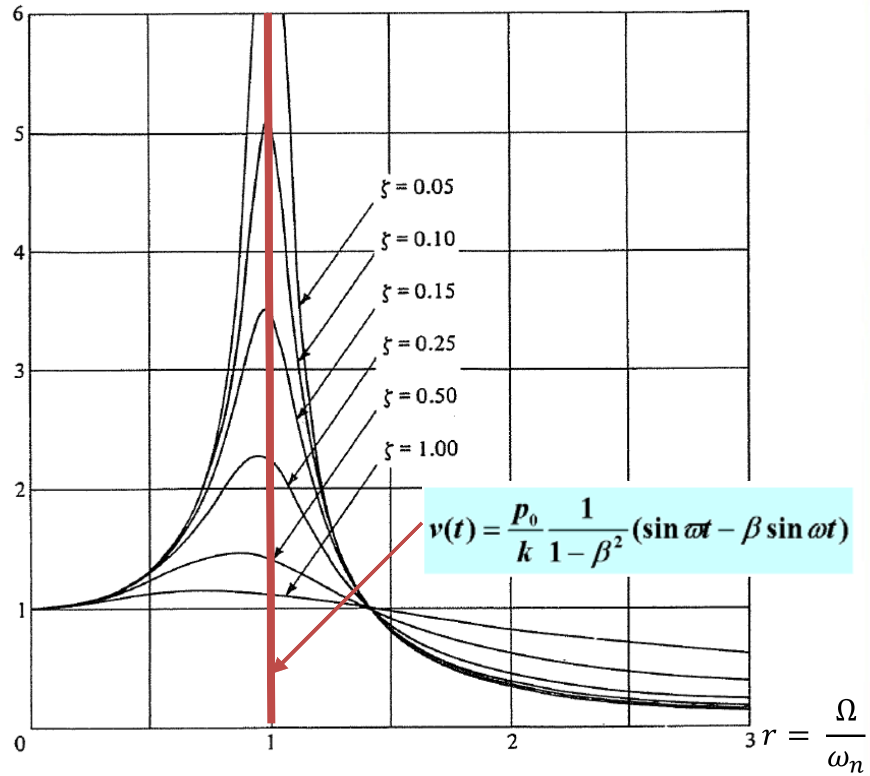
$$\rightarrow u(t) = \frac{U_0}{1-r^2}\cdot[\sin(\Omega t) - r\sin(\omega_n t)]$$

$$\dot{u}(t) = \frac{U_0\Omega}{1-r^2}\cdot\cos(\Omega t) + A_1\,\omega_n\cos(\omega_n t) - A_2\omega_n\sin(\omega_n t)$$

$$\rightarrow \dot{u}(t) = \frac{U_0\Omega}{1-r^2}\cdot\cos(\Omega t) - \frac{U_0 r}{1-r^2}\omega_n\cos(\omega_n t)$$

Frequency-Response Equation

$$H(\Omega) = \frac{U}{U_0} = \frac{1}{1-r^2}$$



$$m\ddot{v} + c\dot{v} + kv = p_0 \sin \varpi t$$

$$\ddot{v} + 2\xi\omega\dot{v} + \omega^2 v = \frac{p_0}{m} \sin \varpi t$$

$$v_c(t) = e^{-\xi\omega t}(A \sin \omega_d t + B \cos \omega_d t)$$

$$v_p(t) = G_1 \sin \varpi t + G_2 \cos \varpi t$$

$$\dot{v}_p(t) = G_1 \varpi \cos \varpi t - G_2 \varpi \sin \varpi t$$

$$\ddot{v}_p(t) = -G_1 \varpi^2 \sin \varpi t - G_2 \varpi^2 \cos \varpi t$$

$$-G_1 \varpi^2 \sin \varpi t - G_2 \varpi^2 \cos \varpi t$$

$$+ 2\xi\omega G_1 \varpi \cos \varpi t - 2\xi\omega G_2 \varpi \sin \varpi t$$

$$+ \omega^2 G_1 \sin \varpi t + \omega^2 G_2 \cos \varpi t = \frac{p_0}{m} \sin \varpi t$$

$$m\ddot{v} + c\dot{v} + kv = p_0 \sin \varpi t$$

$$-G_1\varpi^2 \sin \varpi t - G_2\varpi^2 \cos \varpi t$$
$$+2\xi\omega G_1\varpi \cos \varpi t - 2\xi\omega G_2\varpi \sin \varpi t$$
$$+\omega^2 G_1 \sin \varpi t + \omega^2 G_2 \cos \varpi t = \frac{p_0}{m} \sin \varpi t$$

$$\left(-G_1\varpi^2 - 2G_2\xi\omega\varpi + G_1\omega^2 - \frac{F_0}{m}\right)\sin \varpi t$$
$$= \left(G_2\varpi^2 - 2G_1\xi\omega\varpi - G_2\omega^2\right)\cos \varpi t$$

$$m\ddot{v} + c\dot{v} + kv = p_0 \sin \varpi t$$

$$\begin{cases} -G_1\varpi^2 - 2G_2\xi\omega\varpi + G_1\omega^2 - \dfrac{p_0}{m} = 0 \\ G_2\varpi^2 - 2G_1\xi\omega\varpi - G_2\omega^2 = 0 \end{cases}$$

$$\begin{cases} G_1 = \dfrac{p_0}{k}\dfrac{1-\beta^2}{\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2} \\ G_2 = \dfrac{p_0}{k}\dfrac{-2\xi\beta}{\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2} \end{cases}$$

$$v_p(t) = G_1 \sin \varpi t + G_2 \cos \varpi t$$

$$v_p(t) = \frac{p_0}{k}\frac{1}{\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2}\left[(1-\beta^2)\sin \varpi t - 2\xi\beta \cos \varpi t\right]$$

$$m\ddot{v} + c\dot{v} + kv = p_0 \sin \varpi t$$

$$v(t) = v_c(t) + v_p(t)$$

$$= e^{-\xi\omega t}(A\sin\omega_d t + B\cos\omega_d t)$$

$$+ \frac{p_0}{k}\frac{1}{(1-\beta^2)^2 + (2\xi\beta)^2}\left[(1-\beta^2)\sin \varpi t - 2\xi\beta \cos \varpi t\right]$$

SDOF Harmonic Vibration Damped System

$$m\ddot{u} + c\dot{u} + ku = P_0 \cdot \cos(\Omega t)$$

Particular solution

$$u_p(t) = U \cdot \cos(\Omega t - \alpha)$$

$$\dot{u}_p(t) = -U \cdot \Omega \cdot \sin(\Omega t - \alpha)$$

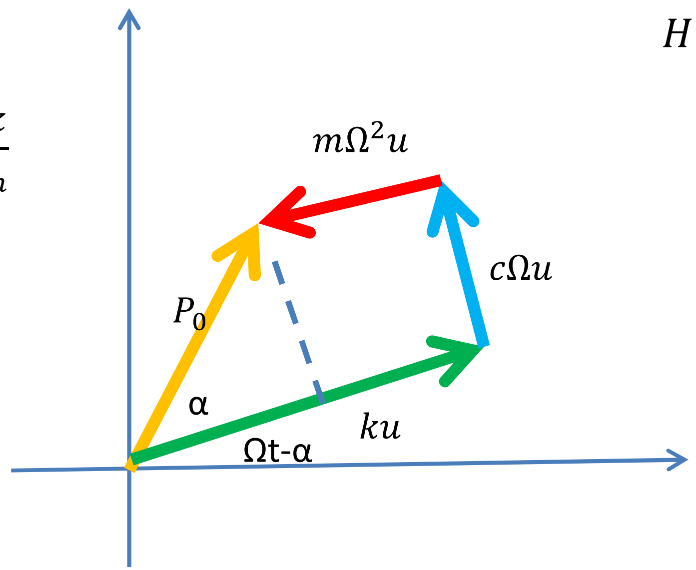
$$\ddot{u}_p(t) = -U \cdot \Omega^2 \cdot \cos(\Omega t - \alpha)$$

$$\rightarrow m(-U \cdot \Omega^2 \cdot \cos(\Omega t - \alpha)) - c\Omega U \cdot \sin(\Omega t - \alpha) + kU \cdot \cos(\Omega t - \alpha) = P_0 \cdot \cos(\Omega t)$$

$$\frac{m}{k} = \frac{1}{\omega_n^2}$$

$$c = \xi c_{cr} = \xi \frac{2k}{\omega_n}$$

$$\frac{c}{k} = \frac{2\xi}{\omega_n}$$



$$A^2 + B^2 = C^2$$

$$U^2(k - m\Omega^2)^2 + (c\Omega U)^2 = {P_0}^2$$

$$\rightarrow U\sqrt{(k - m\Omega^2)^2 + (c\Omega)^2} = P_0$$

$$\rightarrow U = \frac{P_0}{\sqrt{(k - m\Omega^2)^2 + (c\Omega)^2}}$$

Static deflection: $U_0 = \dfrac{P_0}{k}$

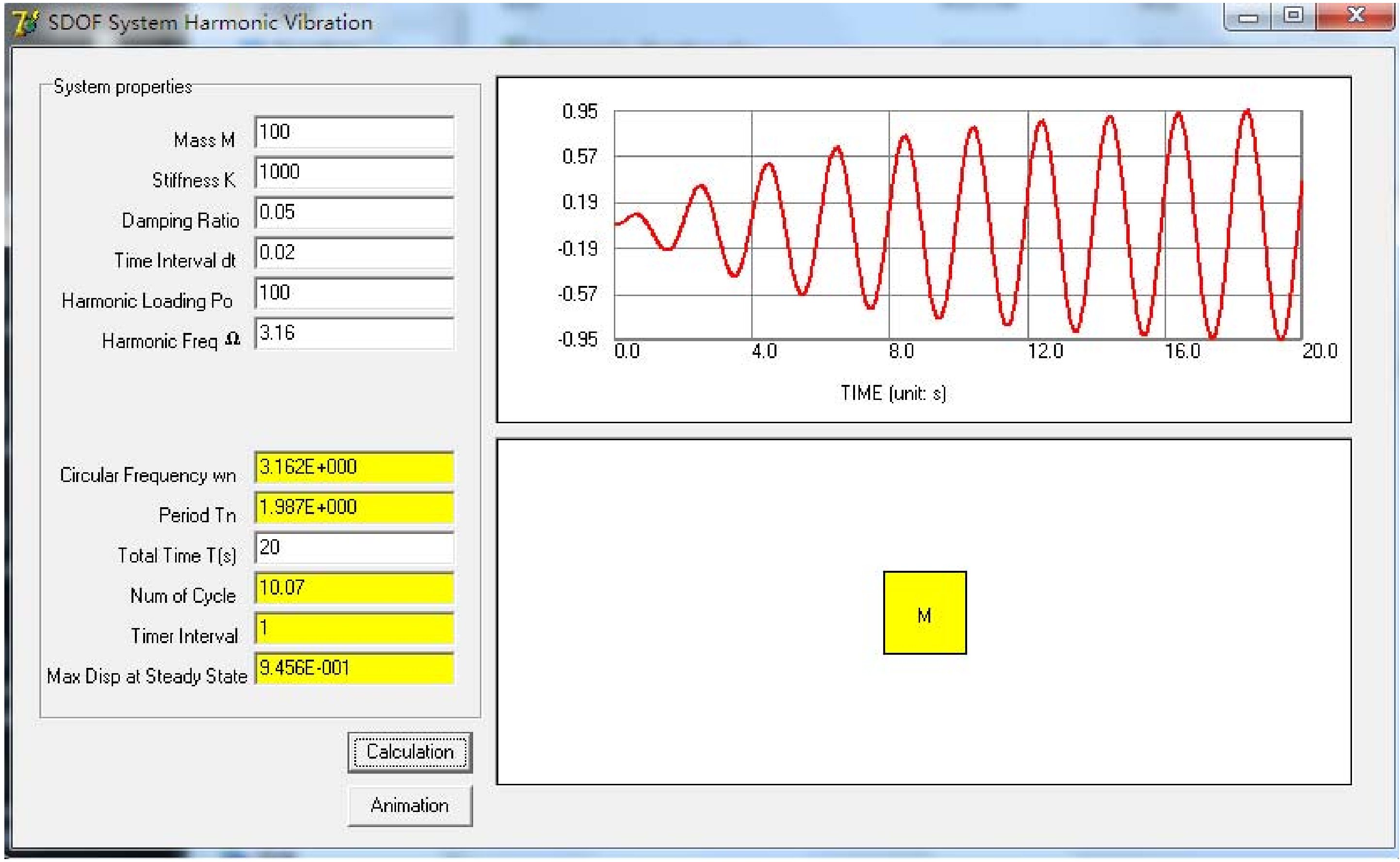
$$H(\Omega) = \frac{U}{U_0}$$

$$= \frac{P_0}{\sqrt{(k - m\Omega^2)^2 + (c\Omega)^2}} \cdot \frac{k}{P_0}$$

$$= \frac{1}{\sqrt{(1 - \frac{m}{k}\Omega^2)^2 + (\frac{c}{k}\Omega)^2}}$$

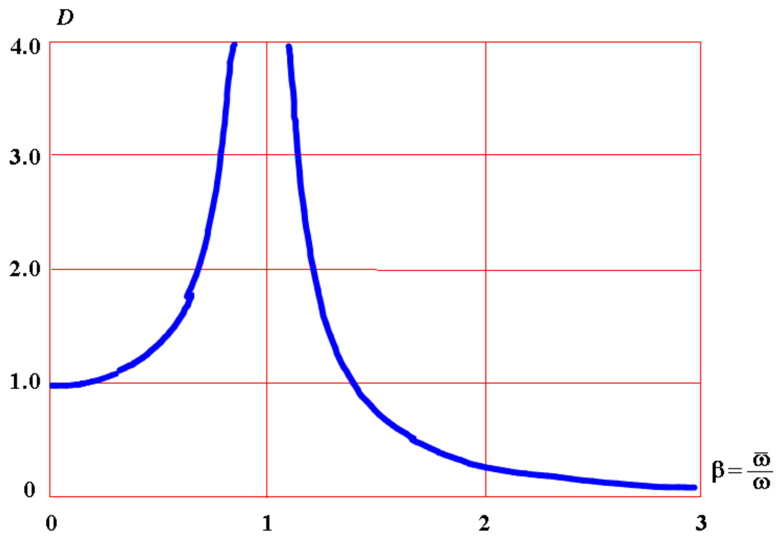
$$= \frac{1}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

Program, SDOF systems due to harmonic Vibration



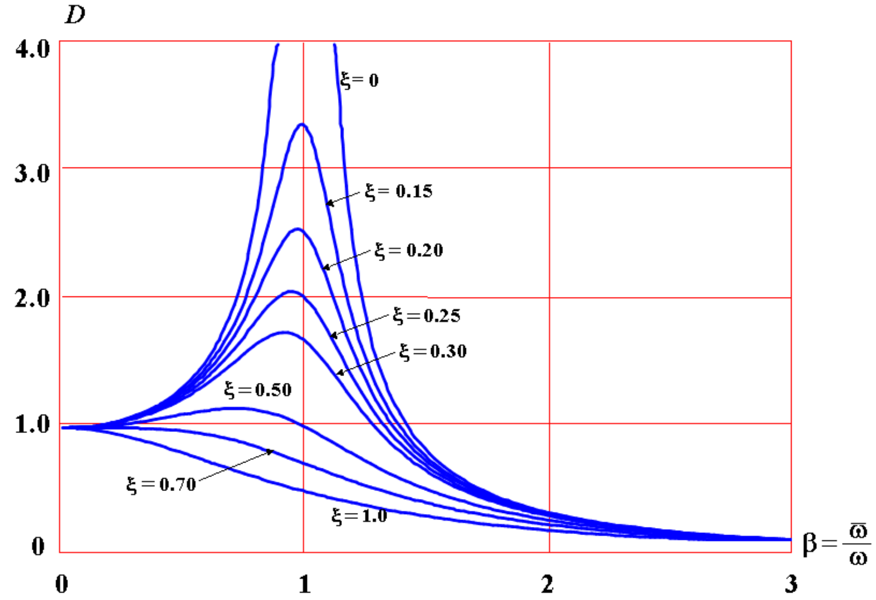
Sympathetic Vibration Situation

$$D = \frac{1}{1-\beta^2} \rightarrow v(t) = \frac{p_0}{k} D(\sin \varpi t - \beta \sin \omega t)$$



Undamped System

$$D = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}}$$



Damped System

Sympathetic Vibration Situation