CIVL7008 Seismic Analysis for Building Structures

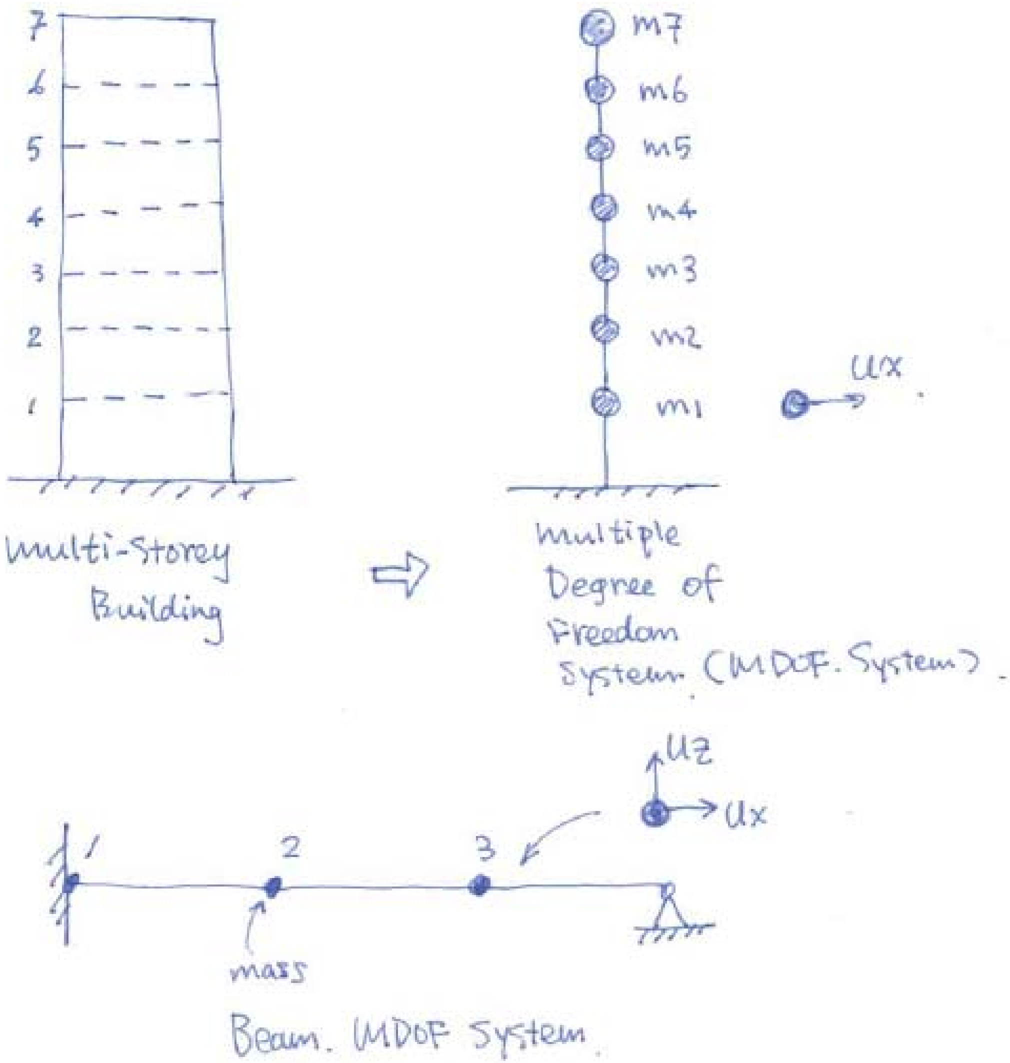
Lec-04 Modal Analysis of MDOF System

University of Hong Kong
Department of Civil Engineering

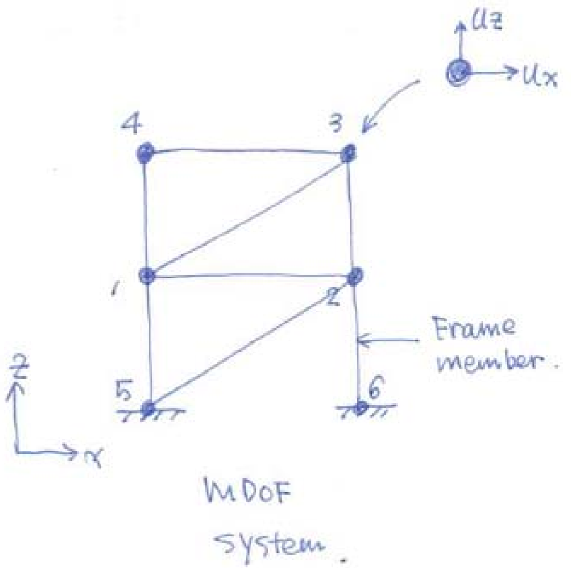
Lec-04 Modal Analysis of MDOF system

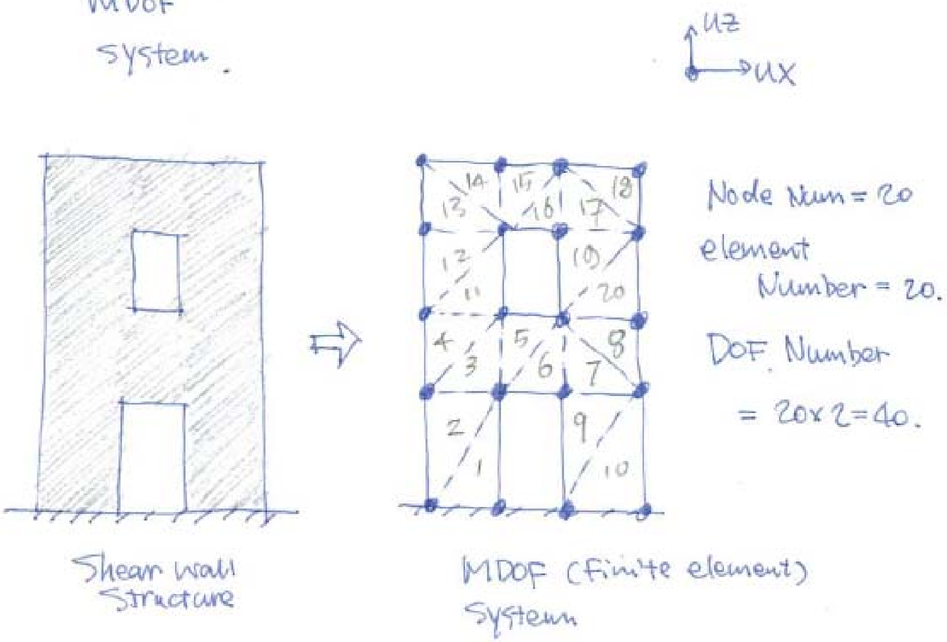
Dr. Dino Chen

What is MDOF System ?

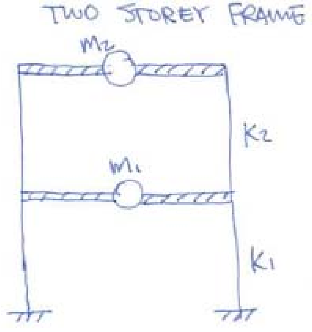


What is MDOF System ?





Modal Analysis of 2 DOF System



① Undamped System + Free Vibration

$$[M]\cdot\{\ddot{u}\}+[K]\cdot\{u\}=\{0\}$$

② Assume Vibration disp.

$$\{u\}=\{\phi\}\cdot\sin\omega t$$

③ $\{\ddot{u}\}=-\omega^2\cdot\{\phi\}\cdot\sin\omega t$

④ $[M]\cdot(-\omega^2)\cdot\{\phi\}\cdot\sin\omega t+[K]\cdot\{\phi\}\sin\omega t=\{0\}$

↳ $([K]-\omega^2\cdot[M])\cdot\{\phi\}=\{0\}$ → Homogeneous system of linear equations.

↳ $\left|[K]-\omega^2\cdot[M]\right|=0$ ← Necessary Condition. For nonzero solution.

↳ $\left|\begin{bmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{bmatrix}-\omega^2\cdot\begin{bmatrix}m_1 & 0\\ 0 & m_2\end{bmatrix}\right|=0$

↳ $\begin{vmatrix}K_{11}-\omega^2 m_1 & K_{12}\\ K_{21} & K_{22}-\omega^2 m_2\end{vmatrix}=0$ ↳ $(K_{11}-\omega^2 m_1)\cdot(K_{22}-\omega^2 m_2)-(K_{21}\times K_{12})=0$ → $\begin{cases}\omega_1\\ \omega_2\end{cases}$

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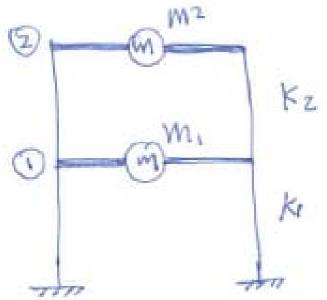
Example.



Storey mass. $m_1 = 60t$ $m_2 = 50t$

Storey lateral stiffness

$K_1 = 5\times10^4$ kN/m.

$K_2 = 3\times10^4$ kN/m.

Compute First & Second Mode period & mode shape.

$$[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad [K] = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}$$

$$([K] - \omega^2 \cdot [M]\cdot)\{\phi\} = 0$$

$$\left| [K] - \omega^2 [M] \right| = 0 \longrightarrow \begin{vmatrix} K_1+K_2 - m_1\omega^2 & -K_2 \\ -K_2 & K_2 - m_2\omega^2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 8\times10^4 - 60\cdot\omega^2 & -3\times10^4 \\ -3\times10^4 & 3\times10^4 - 50\omega^2 \end{vmatrix} = 3\omega^4 - 5.8\times10^3\,\omega^2 + 1.5\times10^6 = 0$$

$$\Rightarrow 3A^2 - 5.8\times10^3 A + 1.5\times10^6 = 0$$

$$\Rightarrow \begin{cases} A_1 = \omega_1^2 = 307.6 & \omega_1 = 17.54 \text{ rad/s} \\ A_2 = \omega_2^2 = 1625.8 & \omega_2 = 40.32 \text{ rad/s} \end{cases}$$

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$$[k] - \omega^2 \cdot [m] \cdot \{\phi\} = \{0\}$$

$$\left(\begin{bmatrix} 8\times10^4 & -3\times10^4 \\ -3\times10^4 & 3\times10^4 \end{bmatrix} - \omega^2 \begin{bmatrix} 60 & 0 \\ 0 & 50 \end{bmatrix} \right) \cdot \{\phi\} = \{0\}$$

$$\omega_1^2 = 307.6$$

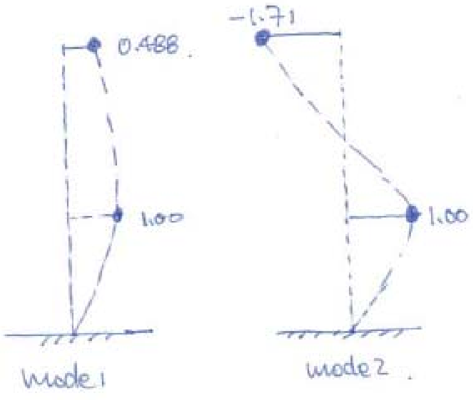
$$\Rightarrow \begin{bmatrix} 8\times10^4 - 307.6\times 60 & -3\times10^4 \\ -3\times10^4 & 3\times10^4 - 50\times 307.6 \end{bmatrix} \{\phi\} = \{0\}$$

$$\Rightarrow \begin{bmatrix} 6.15 & -3 \\ -3 & 1.46 \end{bmatrix} \cdot \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \{0\}$$

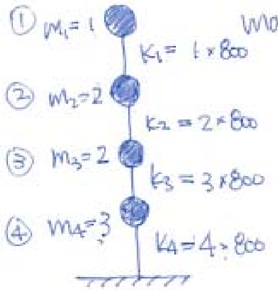
$$\Rightarrow \begin{cases} 6.15x_1 - 3x_2 = 0 \\ -3x_1 + 1.46x_2 = 0 \end{cases} \Rightarrow x_1 = \frac{3x_2}{6.15} = 0.488\, x_2$$

$$\{\phi_1\} = \begin{Bmatrix} 0.488\, x_2 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0.488 \\ 1 \end{Bmatrix} \longrightarrow \text{First mode shape}$$

SAME PROCESS. $\{\phi_2\} = \begin{Bmatrix} -1.71 \\ 1 \end{Bmatrix} \longrightarrow$ Second mode shape



Example. 4-DoF-System. Mode Shape Calculation.



Mass matrix

$$[m] = \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & m_3 & \\ & & & m_4 \end{bmatrix} \qquad [k] = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3+k_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 7 \end{bmatrix} \times 800$$

$$[k] - \omega^2 \cdot [m] = \{0\}$$

$$\hookrightarrow \left| [k] - \omega^2 \cdot [m] \right| = 0 \qquad [m] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\left| [k] - \omega^2 \cdot [m] \right| = \begin{vmatrix} 800-\omega^2 & -800 & 0 & 0 \\ -800 & 2400-2\omega^2 & -1600 & 0 \\ 0 & -1600 & 4000-2\omega^2 & -2400 \\ 0 & 0 & -2400 & 5600-3\omega^2 \end{vmatrix}$$

Determinant:

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

$$= (800-\omega^2) \cdot \begin{vmatrix} 2400-2\omega^2 & -1600 & 0 \\ -1600 & 4000-2\omega^2 & -2400 \\ 0 & -2400 & 5600-3\omega^2 \end{vmatrix} - (-800) \cdot \begin{vmatrix} -800 & -1600 & 0 \\ 0 & 4000-2\omega^2 & -2400 \\ 0 & -2400 & 5600-3\omega^2 \end{vmatrix}$$

$$= (800-\omega^2)\left[(2400-2\omega^2)(4000-2\omega^2)(5600-3\omega^2) - (-2400)(-2400)(2400-2\omega^2) - (-1600)(-1600)(5600-3\omega^2)\right]$$

$$-(-800)\cdot\left[(-800)\cdot(4000-2\omega^2)(5600-3\omega^2) - (-2400)(-2400)(-800)\right] = y$$

$y = f(\omega^2) = 0$; USE EXCEL TO compute ω^2.

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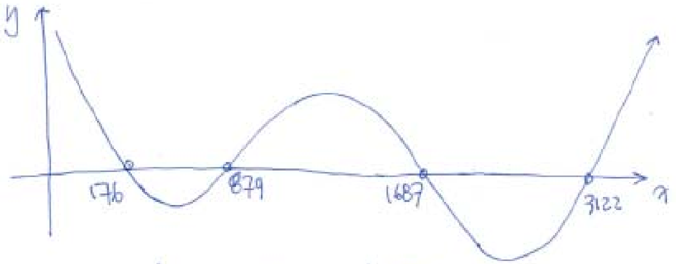
$x = \omega^2$. $y = f(x) = 0$ EXCEL Table Calculation.



$x_1 = 176$	$\omega_1 = 13.2$	$T_1 = 0.474$
$x_2 = 876$	$\omega_2 = 29.65$	$T_2 = 0.212$
$x_3 = 1687$	$\omega_3 = 41.07$	$T_3 = 0.153$
$x_4 = 3122$	$\omega_4 = 55.87$	$T_4 = 0.112$

Mode shape Calculation. $\omega_1^2 = 176$. (Mode shape 1).

$$D(\omega_1^2) = \begin{bmatrix} 800-176 & -800 & 0 & 0 \\ -800 & 2400-352 & -1600 & 0 \\ 0 & -1600 & 4000-352 & -2400 \\ 0 & 0 & -2400 & 5600-528 \end{bmatrix} = \begin{bmatrix} 624 & -800 & 0 & 0 \\ -800 & 2048 & -1600 & 0 \\ 0 & -1600 & 3648 & -2400 \\ 0 & 0 & -2400 & 5072 \end{bmatrix}$$

$$D_{bb} = \begin{bmatrix} 2048 & -1600 & 0 \\ -1600 & 3648 & -2400 \\ 0 & -2400 & 5072 \end{bmatrix} \qquad D_{ba} = \begin{Bmatrix} -800 \\ 0 \\ 0 \end{Bmatrix}$$

$$D_{bb}^{-1} = \frac{1}{12805312} \begin{bmatrix} 12444 & 7925 & 3750 \\ 7925 & 10144 & 4800 \\ 3750 & 4800 & 4796 \end{bmatrix}$$

$$\{\phi_1\}_r = -[D_{bb}]^{-1} \cdot D_{ba}$$

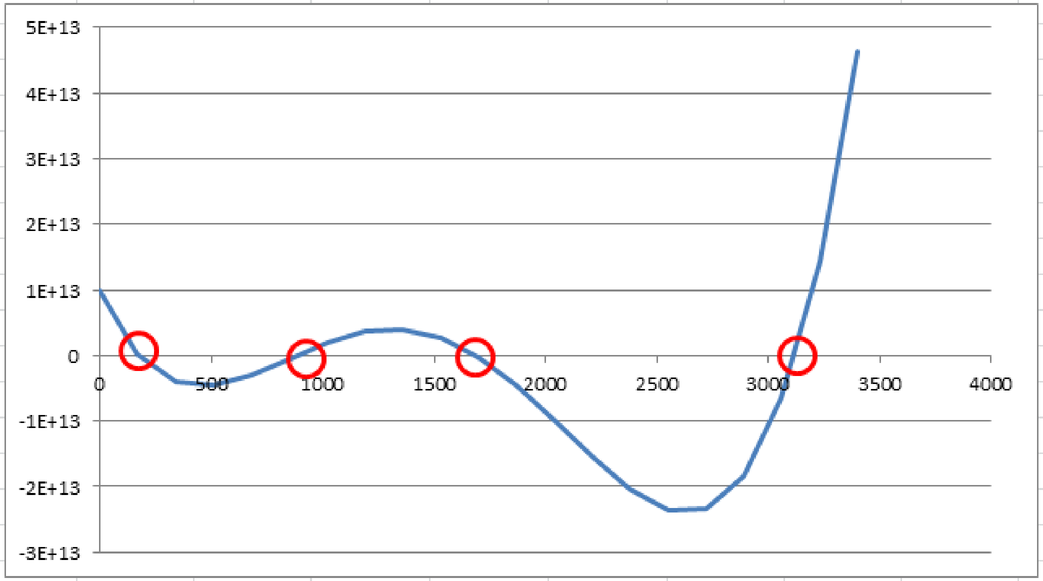
$$= -\frac{1}{12865312} \times \begin{Bmatrix} 12444 \times (-800) \\ 7925 \times (-800) \\ 3750 \times (-800) \end{Bmatrix} = \begin{Bmatrix} 0.777 \\ 0.495 \\ 0.234 \end{Bmatrix}$$

$$\Rightarrow \phi_1 = \begin{Bmatrix} 1.000 \\ 0.777 \\ 0.495 \\ 0.234 \end{Bmatrix}$$

Use Excel table to Eigenvalue

start	0
inc	170

step	x	y
0	0	9.8304E+12
1	170	2.56339E+11
2	340	-3.96523E+12
3	510	-4.54876E+12
4	680	-2.96814E+12
5	850	-4.56725E+11
6	1020	1.99265E+12
7	1190	3.62772E+12
8	1360	3.93675E+12
9	1530	2.64852E+12
10	1700	-2.676E+11
11	1870	-4.60174E+12
12	2040	-9.90348E+12
13	2210	-1.54819E+13
14	2380	-2.04053E+13
15	2550	-2.35019E+13
16	2720	-2.3359E+13
17	2890	-1.83234E+13
18	3060	-6.50158E+12
19	3230	1.42408E+13
20	3400	4.62784E+13



mode shape 2.

$\omega_2^2 = 876$

$$D(\omega_2) = \begin{bmatrix} -76 & -800 & 0 & 0 \\ -800 & 648 & -1600 & 0 \\ 0 & -1600 & 2248 & -2400 \\ 0 & 0 & -2400 & 2972 \end{bmatrix} = \begin{bmatrix} D_{aa} & D_{ab} \\ D_{ba} & D_{bb} \end{bmatrix}$$

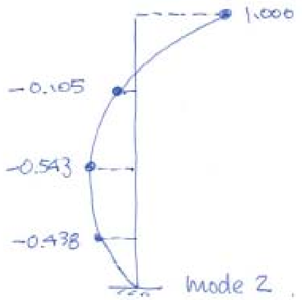
$$D_{bb} = \begin{bmatrix} 648 & -1600 & 0 \\ -1600 & 2248 & -2400 \\ 0 & -2400 & 2972 \end{bmatrix} \qquad D_{ab} = \begin{bmatrix} -800 \\ 0 \\ 0 \end{bmatrix}$$

$$D_{bb}^{-1} = \frac{1}{219108616} \begin{bmatrix} -28783 & -148600 & -120000 \\ -148600 & -60183 & -48600 \\ -120000 & -48600 & 34478 \end{bmatrix}$$

$$(\phi_2)_r = -D_{bb}^{-1} \cdot D_{ab} = \frac{1}{219108616} \times (-800) \times \begin{Bmatrix} -28783 \\ -148600 \\ -120000 \end{Bmatrix} = \begin{Bmatrix} -0.105 \\ -0.543 \\ -0.438 \end{Bmatrix}$$

$$\phi_2 = \begin{Bmatrix} 1.000 \\ -0.105 \\ -0.543 \\ -0.438 \end{Bmatrix}$$

mode shape 2

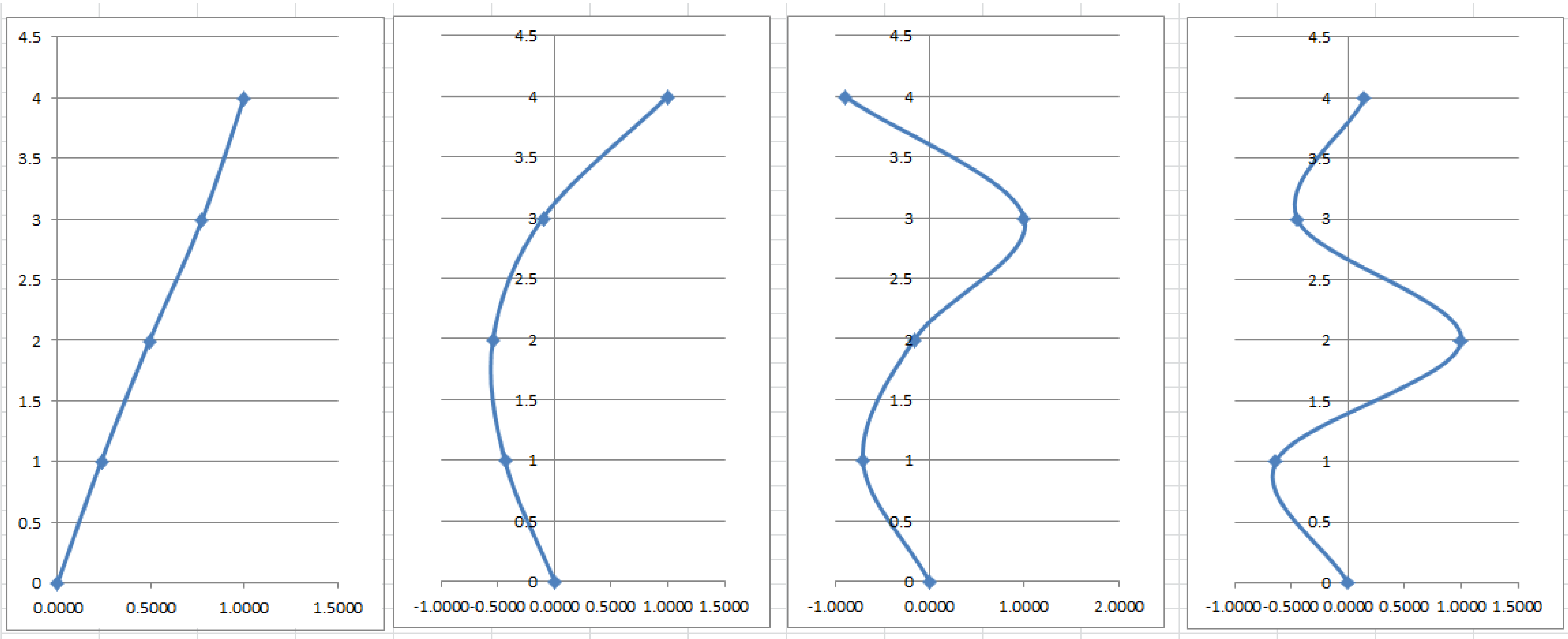


mode shape matrix

$$\phi = \begin{bmatrix} 1.000 & 1.000 & -0.90145 & 0.15436 \\ 0.79910 & -0.09963 & 1.000 & -0.44817 \\ 0.44655 & -0.53989 & -0.15859 & 1.0000 \\ 0.24606 & -0.43761 & -0.70797 & -0.63688 \end{bmatrix}$$

↑ mode 1 ↑ mode 2 ↑ mode 3 ↑ mode 4.

Use Excel to Plot Mode Shape



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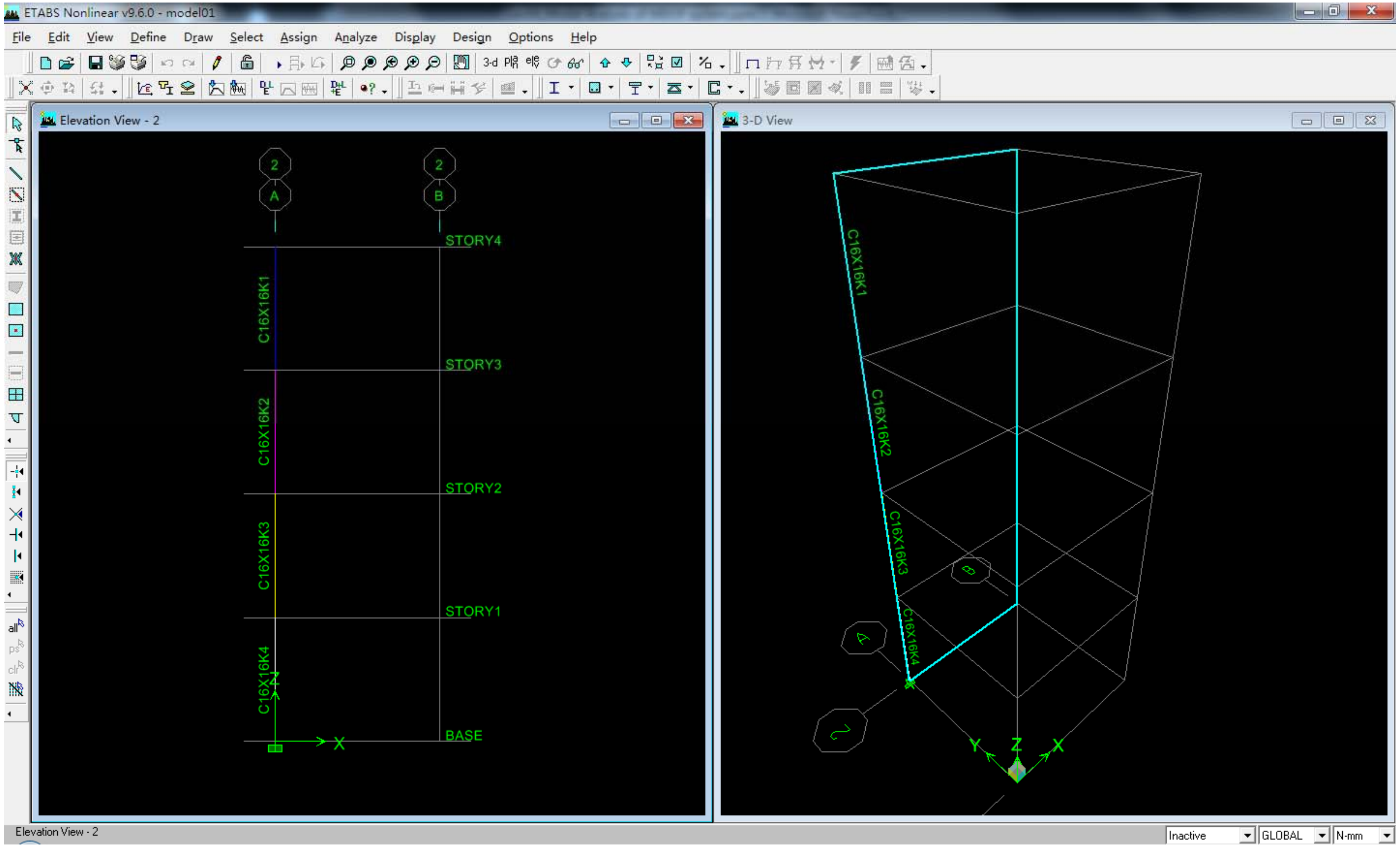
Etabs or SAP2000 model compute mode shape



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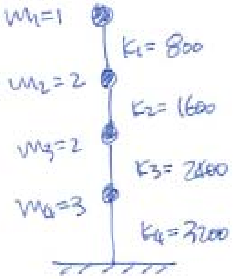


Shear Stiffness.

$$K=\frac{GA_s}{L}$$

$L = 3000$ (mm)

$G = 10416.67$ MPa

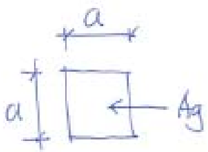
$A_s = 0.8333\,A_g$

$A_g = a^2$

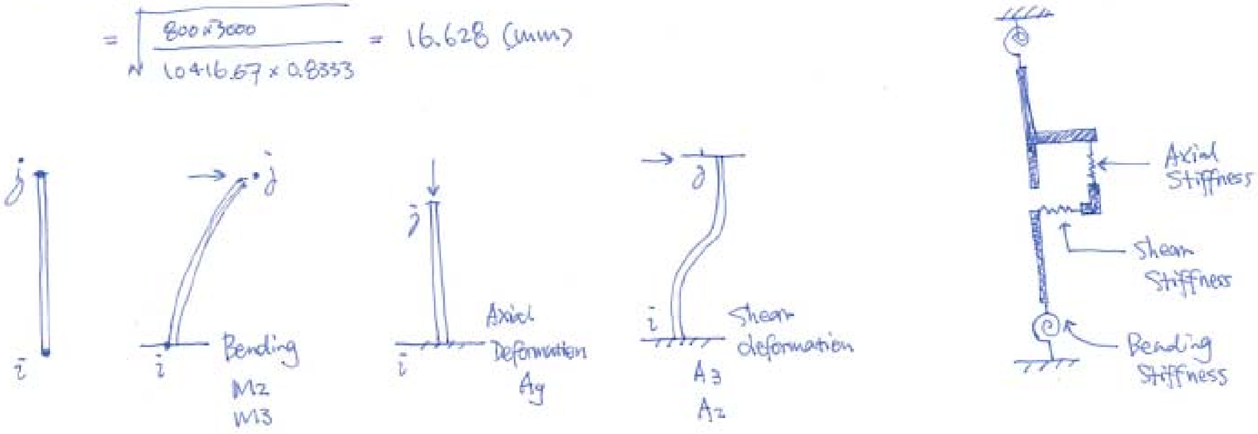
$K_1 = 800$ (N/mm)

$$a=\sqrt{A_g}=\sqrt{\frac{KL}{G}\times\frac{1}{0.8333}}$$

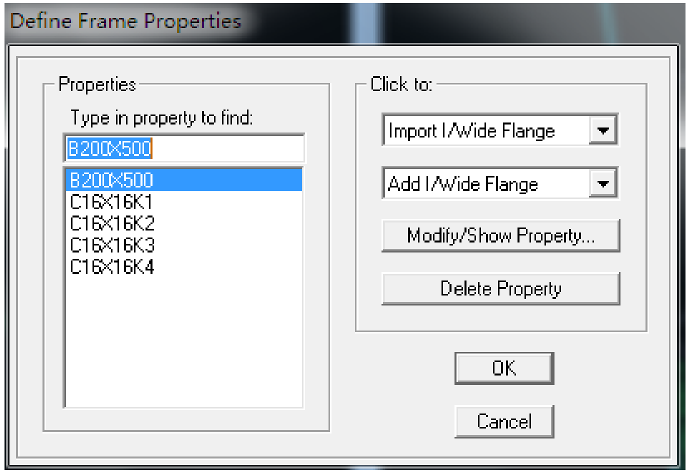
$$=\sqrt{\frac{800\times3000}{10416.67\times0.8333}}=16.628\ \text{(mm)}$$

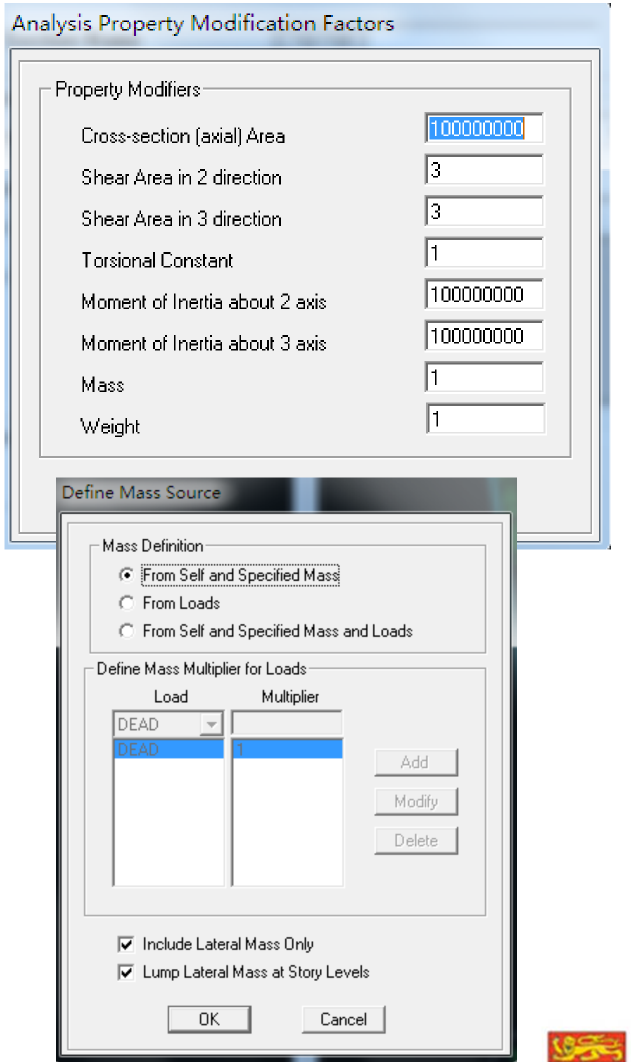


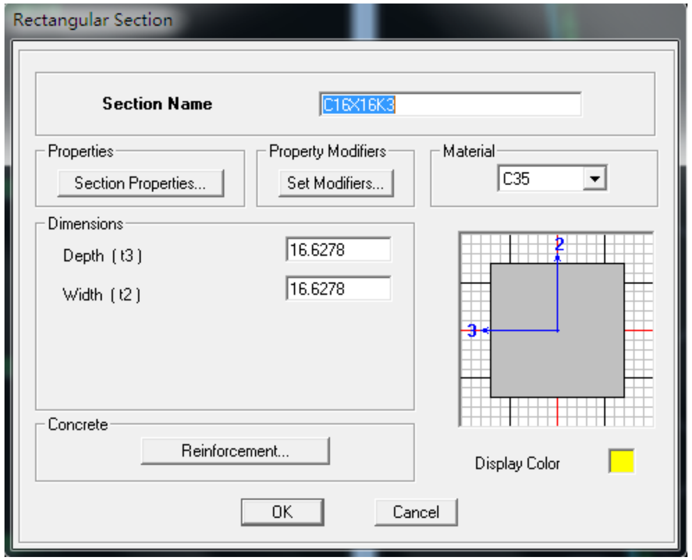
- Axial Area A_g $=\times10^8$
- Shear Area A_3 $=\times1$
- Shear Area A_2 $=\times1$
- Torsional Constant T
- Moment of Inertia About Axis 2 $=\times10^8$
- Moment of Inertia About Axis 3 $=\times10^8$



Etabs or SAP2000 model compute mode shape







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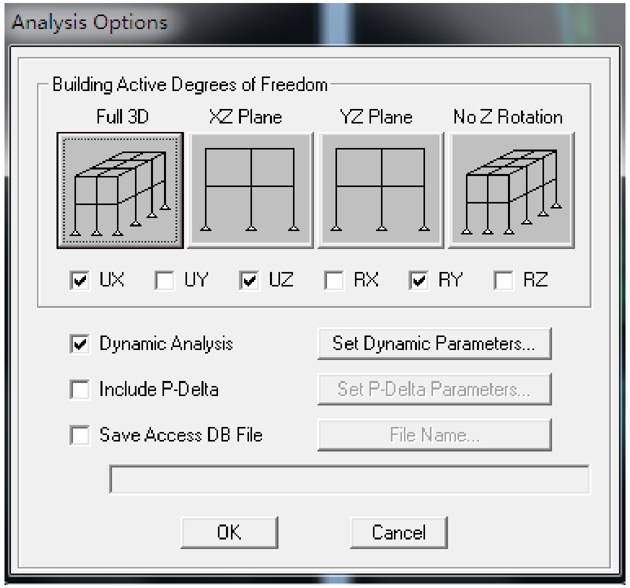

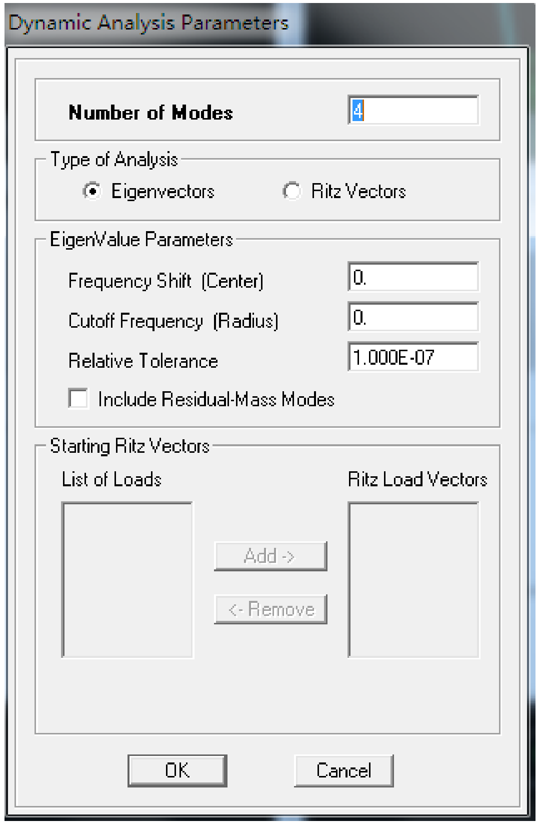

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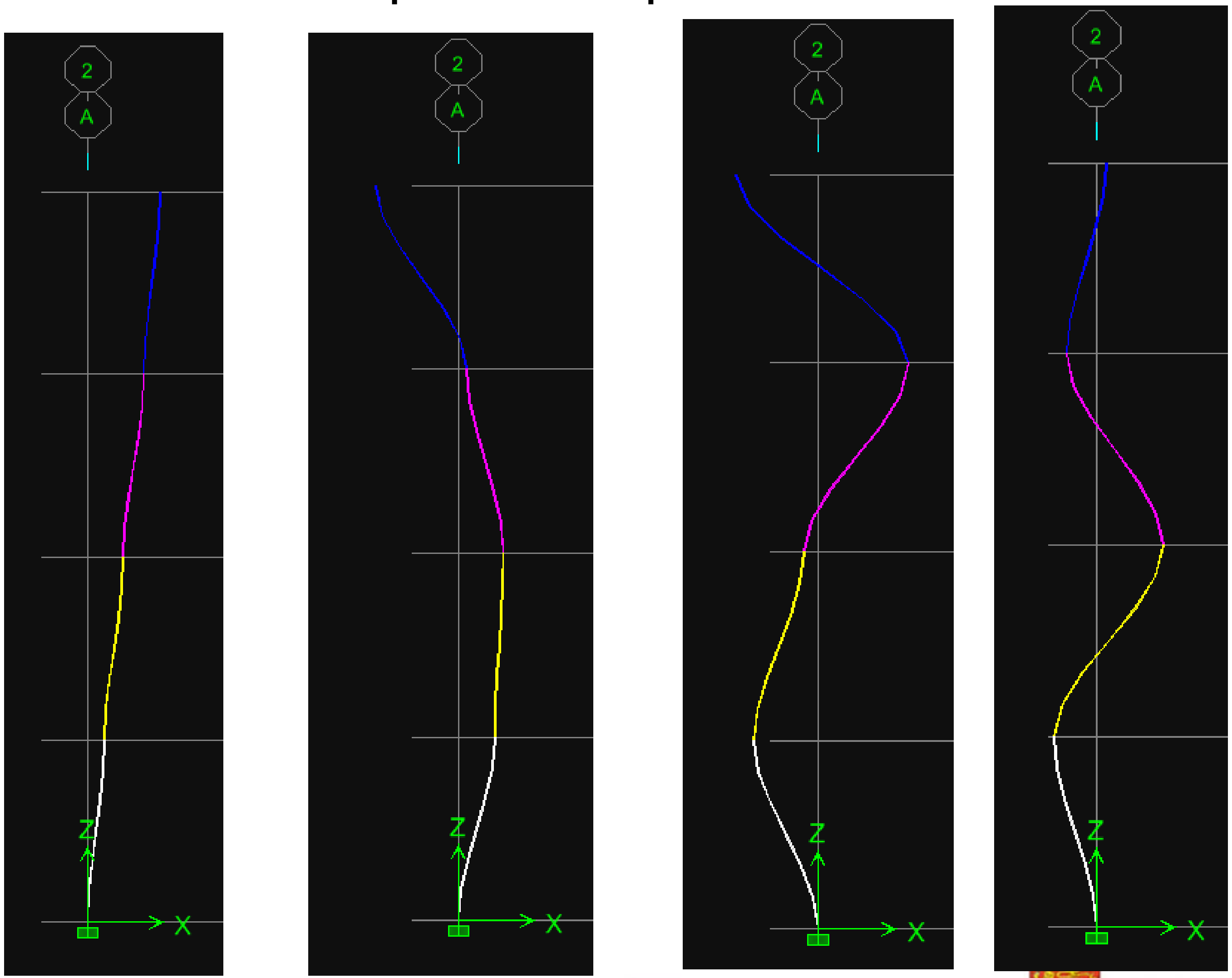
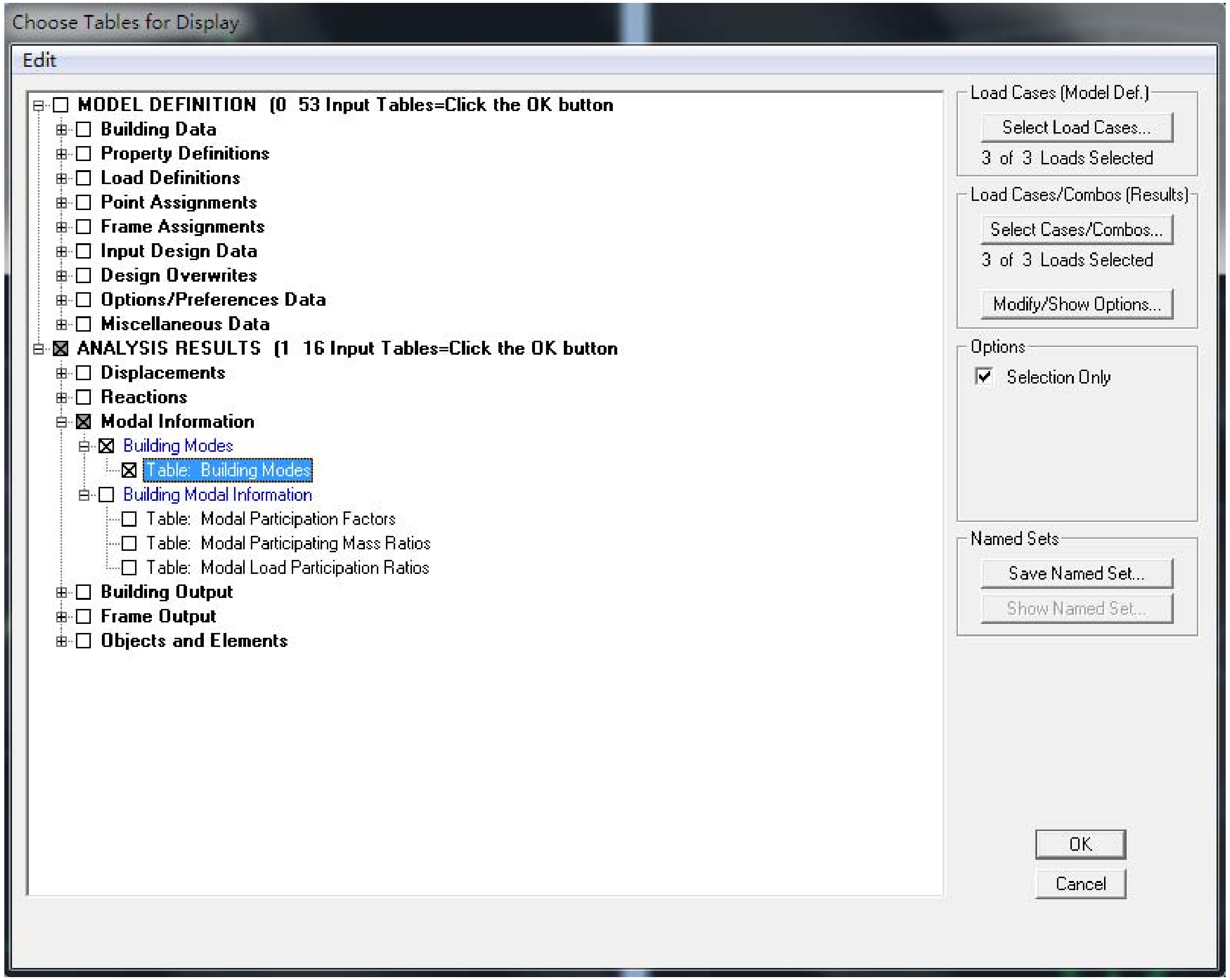


Etabs or SAP2000 model compute mode shape



Lec-04 Modal Analysis of MDOF system

Etabs or SAP2000 model compute mode shape

Building Modes

Edit View

Building Modes

Story	Diaphragm	Mode	UX	UY	UZ	RX	RY	RZ
STORY4	D1	1	0.5917	0.0000	0.0000	0.00000	0.00000	0.00000
STORY3	D1	1	0.4596	0.0000	0.0000	0.00000	0.00000	0.00000
STORY2	D1	1	0.2921	0.0000	0.0000	0.00000	0.00000	0.00000
STORY1	D1	1	0.1376	0.0000	0.0000	0.00000	0.00000	0.00000
STORY4	D1	2	-0.6758	0.0000	0.0000	0.00000	0.00000	0.00000
STORY3	D1	2	0.0685	0.0000	0.0000	0.00000	0.00000	0.00000
STORY2	D1	2	0.3667	0.0000	0.0000	0.00000	0.00000	0.00000
STORY1	D1	2	0.2972	0.0000	0.0000	0.00000	0.00000	0.00000
STORY4	D1	3	-0.4321	0.0000	0.0000	0.00000	0.00000	0.00000
STORY3	D1	3	0.4783	0.0000	0.0000	0.00000	0.00000	0.00000
STORY2	D1	3	-0.0756	0.0000	0.0000	0.00000	0.00000	0.00000
STORY1	D1	3	-0.3388	0.0000	0.0000	0.00000	0.00000	0.00000
STORY4	D1	4	0.0808	0.0000	0.0000	0.00000	0.00000	0.00000
STORY3	D1	4	-0.2351	0.0000	0.0000	0.00000	0.00000	0.00000
STORY2	D1	4	0.5239	0.0000	0.0000	0.00000	0.00000	0.00000
STORY1	D1	4	-0.3336	0.0000	0.0000	0.00000	0.00000	0.00000

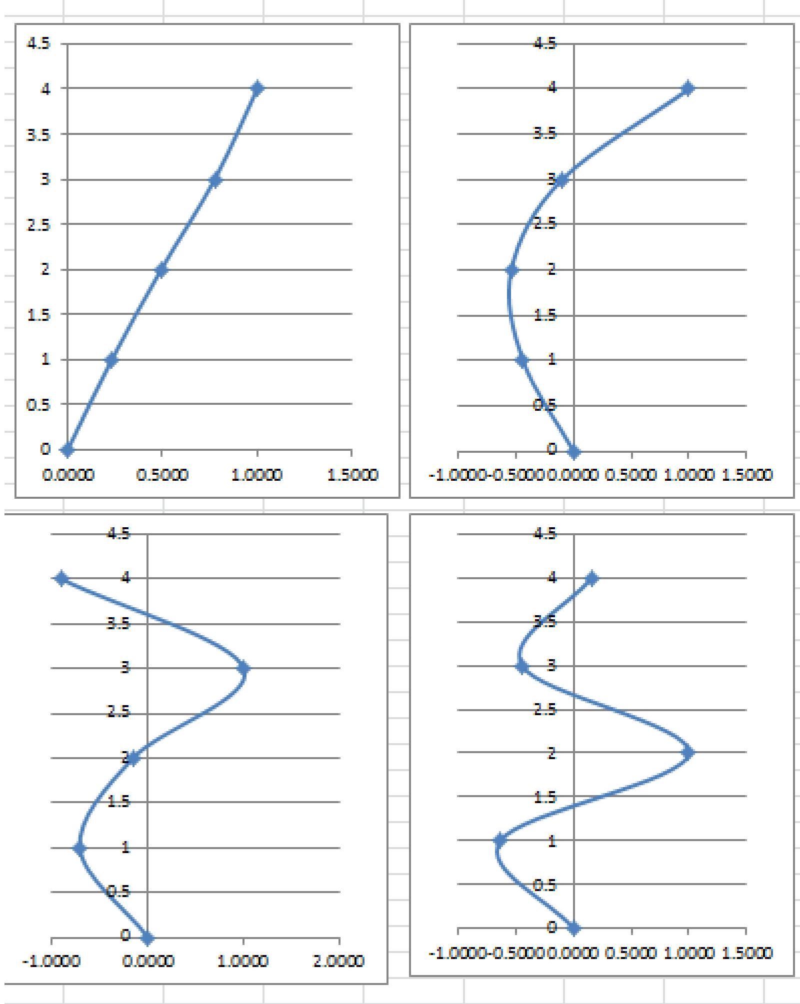
OK

Normalization

MODE	1	2	3	4
STORY4	0.5917	-0.6758	-0.4321	0.0808
STORY3	0.4596	0.0685	0.4783	-0.2351
STORY2	0.2921	0.3667	-0.0756	0.5239
STORY1	0.1376	0.2972	-0.3388	-0.3336

max	0.5917	0.3667	0.4783	0.5239
min	0.1376	-0.6758	-0.4321	-0.3336
absmax	0.5917	0.6758	0.4783	0.5239
sign	1	-1	1	1
maxv	0.5917	-0.6758	0.4783	0.5239

Storey	mode 1	mode 2	mode 3	mode 4
4	1.0000	1.0000	-0.9034	0.1542
3	0.7767	-0.1014	1.0000	-0.4487
2	0.4937	-0.5426	-0.1581	1.0000
1	0.2326	-0.4398	-0.7083	-0.6368
0	0.0000	0.0000	0.0000	0.0000



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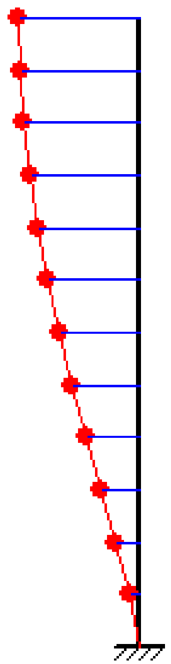
How about more than 4 storey Building (Simplified model) ?



Transform the Matrix D to Matrix B

$$[M]\{\ddot{x}(t)\} + [K]\{\dot{x}(t)\} = \{0\}$$

Assume $[M]^{-1}[K] = \{P\}$

$$\left.\begin{aligned}([P] - \omega^2[I])\{X\} &= \{0\} \\ [P]\{X\} &= \omega^2\{X\}\end{aligned}\right\}$$

$$\begin{vmatrix} P_{11} - \omega^2 & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} - \omega^2 & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} - \omega^2 \end{vmatrix} = 0$$

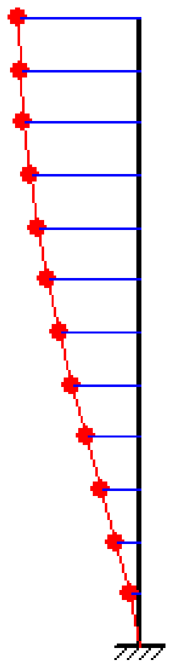
Take

$$\{Z\} = \left[\sqrt{M}\right]\{X\}$$

$$[B]\{Z\} = \omega^2\{Z\}$$

$$[B] = \left[\sqrt{M}\right]^{-1}[K]\left[\sqrt{M}\right]^{-1}$$

How about more than 4 storey Building (Simplified model) ?

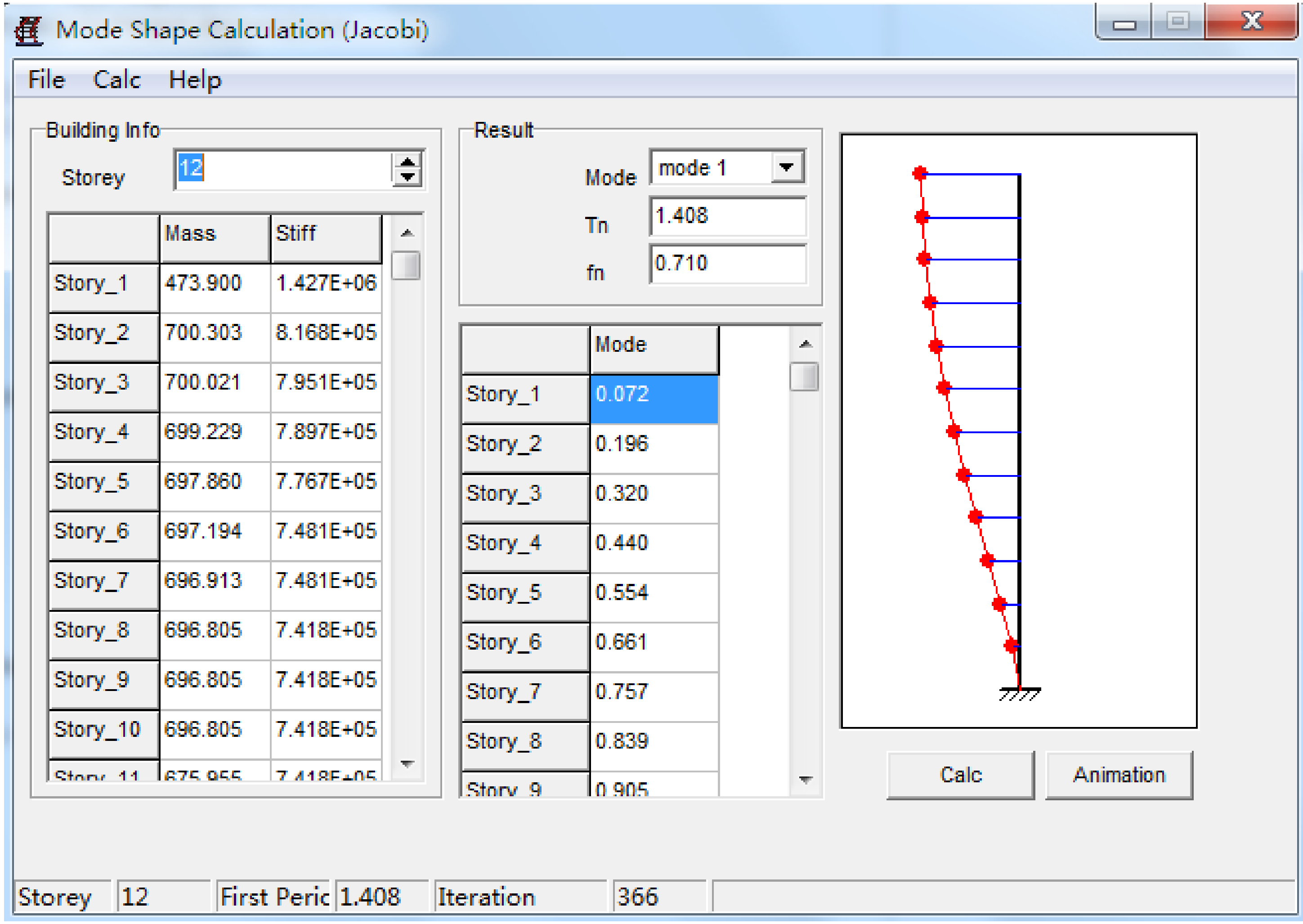


$$[B]=\begin{bmatrix} \frac{K_1+K_2}{m_1} & \frac{-K_2}{\sqrt{m_1}\sqrt{m_2}} & 0 & \cdots & 0 \\ \frac{-K_2}{\sqrt{m_1}\sqrt{m_2}} & \frac{K_2+K_3}{m_2} & \frac{-K_3}{\sqrt{m_2}\sqrt{m_3}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \frac{-K_{n-1}}{\sqrt{m_{n-2}}\sqrt{m_{n-1}}} & \frac{K_{n-1}+K_n}{m_{n-1}} & \frac{-K_n}{\sqrt{m_{n-1}}\sqrt{m_n}} \\ 0 & 0 & \cdots & \frac{-K_n}{\sqrt{m_{n-1}}\sqrt{m_n}} & \frac{K_n}{m_n} \end{bmatrix}$$

Relationship between {x} and {z}, which use to compute mode shape

$$\{X\}=\left[\sqrt{M}\right]^{-1}\{Z\}$$

Mode Shape Calculation Program



Mode Shape Calculation Program

```
//form matrix K and Matrix M
for i:=1 to n do
begin
 k[i]:=strtofloat(stringgrid1.Cells[2,i]);
 m[i]:=strtofloat(stringgrid1.Cells[1,i]);
end;
//Form Matrix B
 setlength(b,n+1,n+1);
 setlength(v,n+1,n+1);
 setlength(m1,n+1,n+1);
 setlength(rr,n+1,n+1);
 setlength(rr2,n+1,n+1);
 for i:=1 to n do
 for j:=1 to n do
  begin
  b[i,j]:=0;  //Clear
  end;
 for i:=2 to n-1 do
 begin
  b[i,i]:=(k[i]+k[i+1])/m[i];
  b[i,i-1]:=-k[i]/(sqrt(m[i-1])*sqrt(m[i]));
  b[i,i+1]:=-k[i+1]/(sqrt(m[i])*sqrt(m[i+1]));
 end;
  b[1,1]:=(k[1]+k[2])/m[1];
  b[1,2]:=-k[2]/(sqrt(m[1])*sqrt(m[2]));
  b[n,n]:=k[n]/m[n];
  b[n,n-1]:=-k[n]/(sqrt(m[n-1])*sqrt(m[n]));
  //Form Matrix B
  //
  combobox1.Clear;
  for i:=1 to n do
  begin
  combobox1.Items.Add(format('mode %d',[i]));
  end;
```

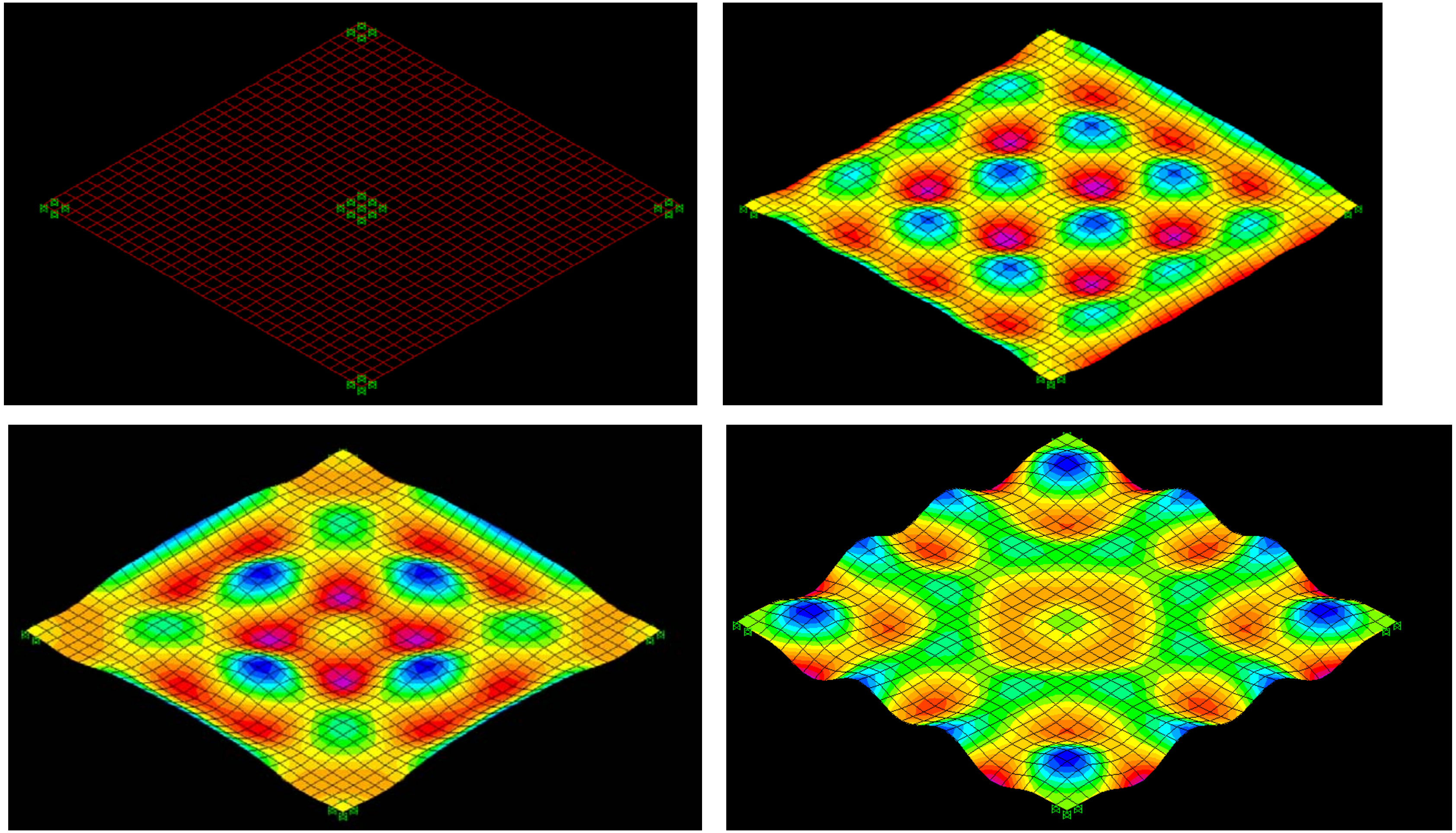
```
jacobi(b,n,d,v,nrot);


statusbar1.Panels[5].Text:=inttostr(nrot);
for i:=1 to n do
zx[i]:=2*pi/sqrt(d[i]);
for i:=1 to n do
num[i]:=i;
for j:=1 to n do
for i:=1 to n-j do
begin
 if zx[i]<zx[i+1] then
 begin
 temp:=zx[i];
 temp2:=num[i];
 zx[i]:=zx[i+1];
 num[i]:=num[i+1];
 zx[i+1]:=temp;
 num[i+1]:=temp2;
 end;
end;  //order
```

$$[B]=\begin{bmatrix} \frac{K_1+K_2}{m_1} & \frac{-K_2}{\sqrt{m_1}\sqrt{m_2}} & 0 & \cdots & 0 \\ \frac{-K_2}{\sqrt{m_1}\sqrt{m_2}} & \frac{K_2+K_3}{m_2} & \frac{-K_3}{\sqrt{m_2}\sqrt{m_3}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \frac{-K_{n-1}}{\sqrt{m_{n-2}}\sqrt{m_{n-1}}} & \frac{K_{n-1}+K_n}{m_{n-1}} & \frac{-K_n}{\sqrt{m_{n-1}}\sqrt{m_n}} \\ 0 & 0 & \cdots & \frac{-K_n}{\sqrt{m_{n-1}}\sqrt{m_n}} & \frac{K_n}{m_n} \end{bmatrix}$$

Lec-04 Modal Analysis of MDOF system

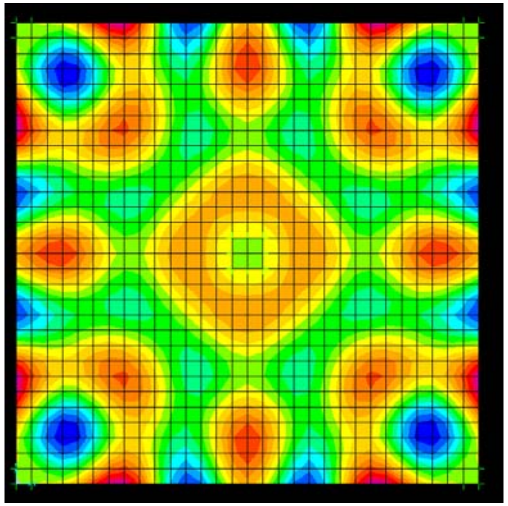
Dr. Dino Chen

Mode Shape Analysis of Shell Element (High Freq mode)

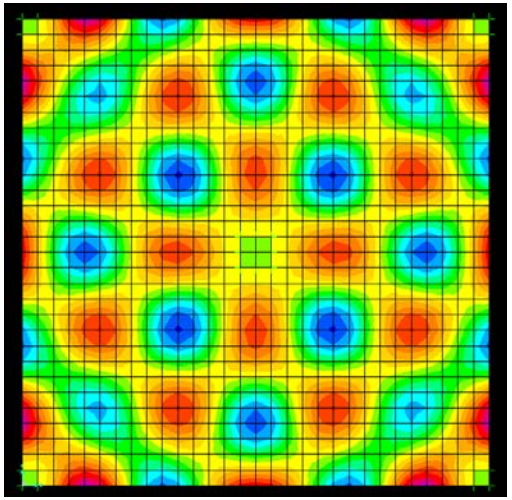


Lec-04 Modal Analysis of MDOF system

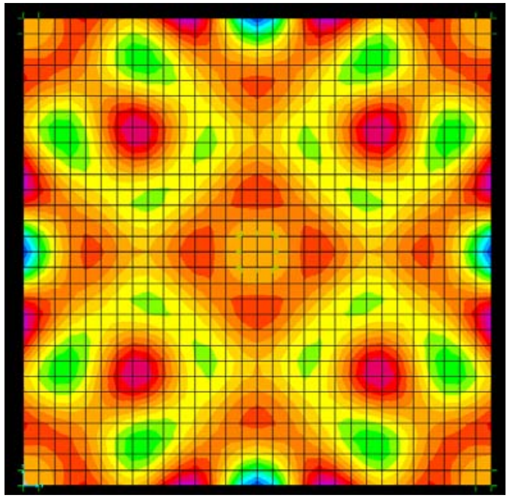
Dr. Dino Chen



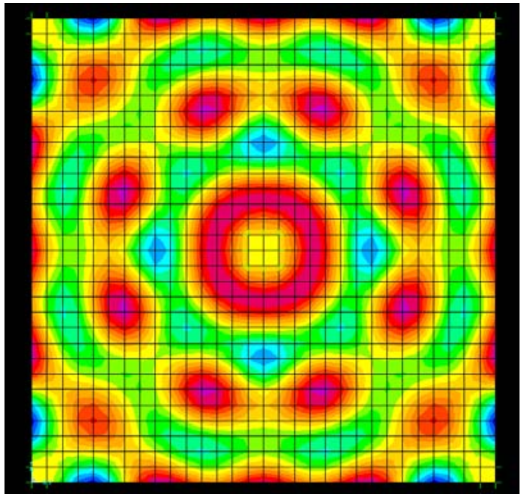
Mode 59 - Period 0.01466



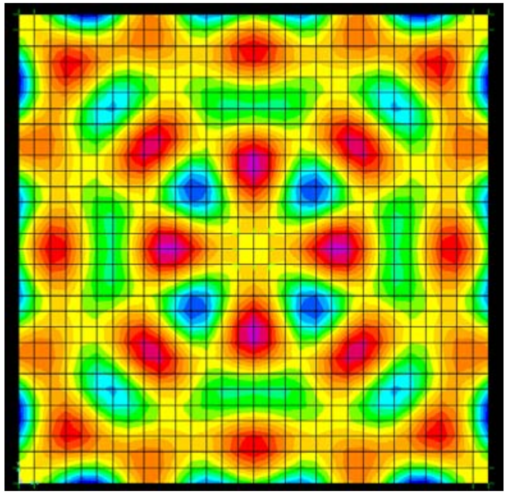
Mode 62 - Period 0.01365



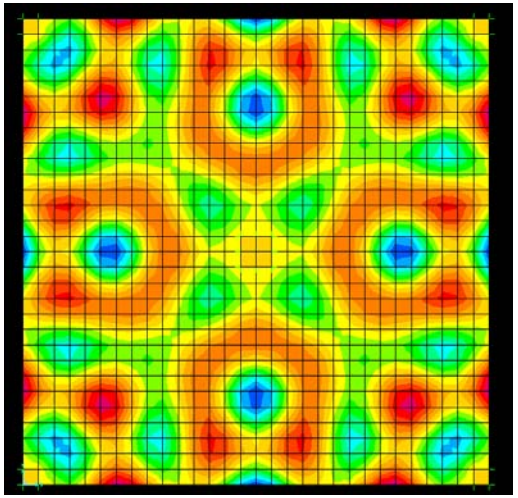
Mode 68 - Period 0.01241



Mode 84 - Period 0.01003



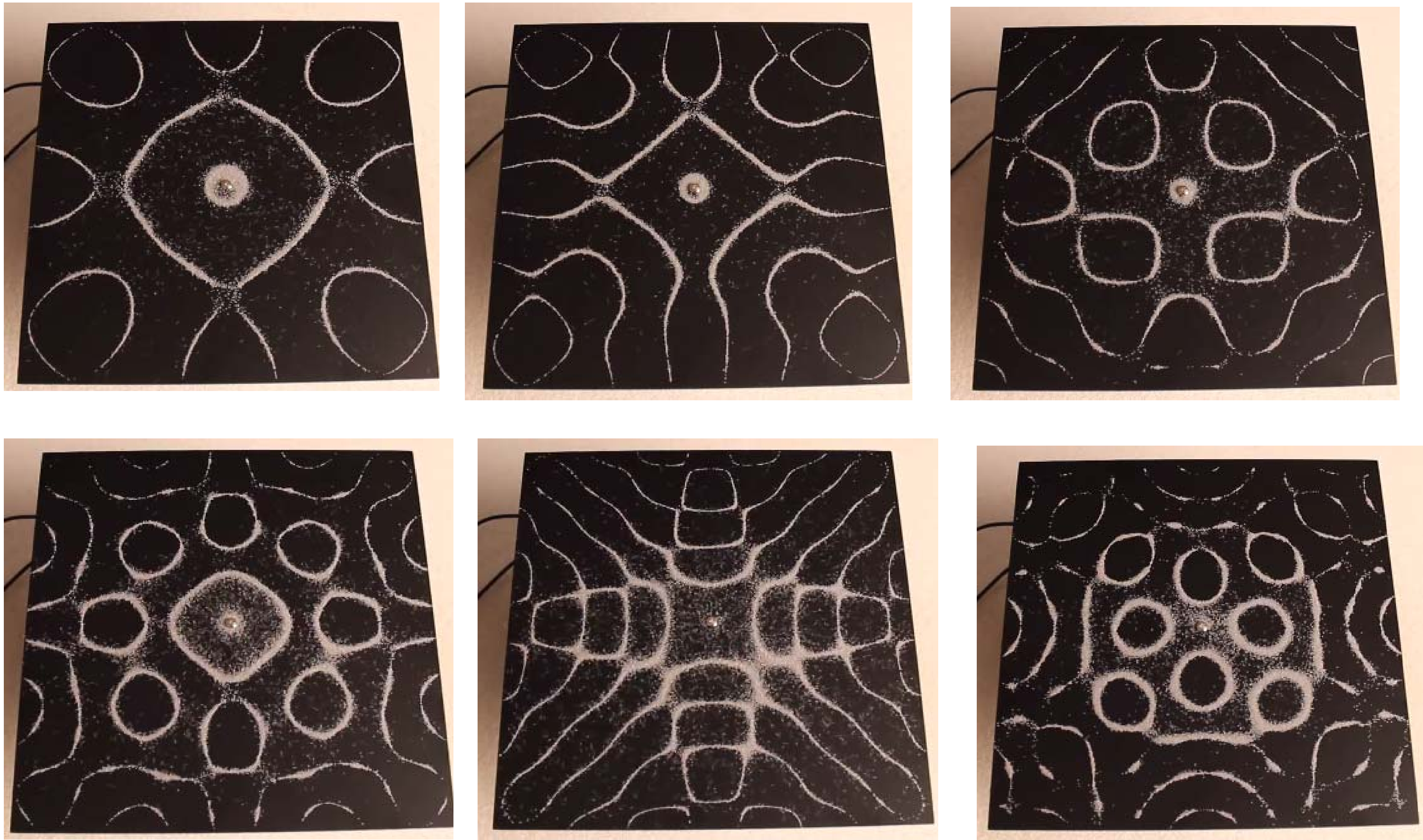
Mode 85 - Period 0.00980



Mode 93 - Period 0.00917

Lec-04 Modal Analysis of MDOF system

Dr. Dino Chen



Lec-04 Modal Analysis of MDOF System