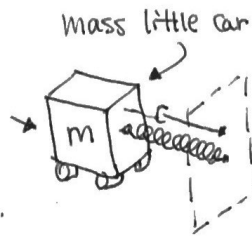


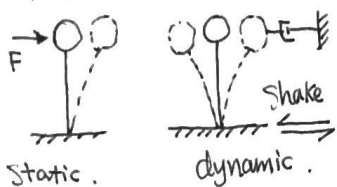
Arrangement of the course.

- Lesson 1. Introduction of the course.
 - Lesson 2. Vibration of SDOF System.
 - Lesson 3. Forced Vibration of SDOF system.
 - Lesson 4. Modal Analysis of MDOF system.
 - Lesson 5. Vibration of MDOF system.
 - Lesson 6. Spectrum Analysis method of MDOF system.
- Basic Theory. hand Calculation + make program (Pascal).



PART 1

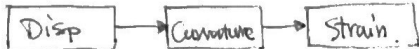
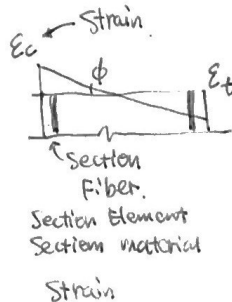
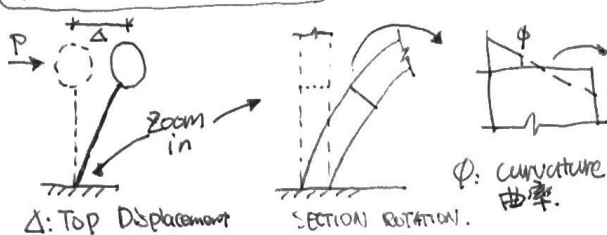
Introduction



Static.
Equation of Motion
 $m\ddot{u} + c\dot{u} + ku = P(t)$
If $P(t) = \text{const}$, $\ddot{u} = 0$, $\dot{u} = 0$
→ $ku = P = \text{const}$

Every Thing is DYNAMIC

Generalized Displacement



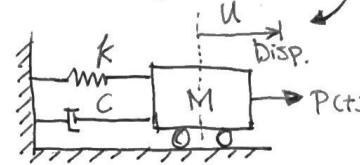
Generalized Displacement

DISP → Section Deformation
inter-Force → Curvature ϕ → Mat Strain
→ Section Force → mat Stress

Generalized Inter-Force. $[K][u] = [P]$

Dynamic System

SDOF = Single Degree of Freedom.

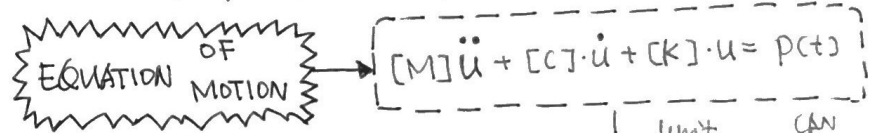


CAR SYSTEM IMAGE

$[K]$ Stiffness. Spring
 $[c]$ damper. Viscous Damper
 $[m]$ mass.

SDOF system.

$[P]$ external Force time-history

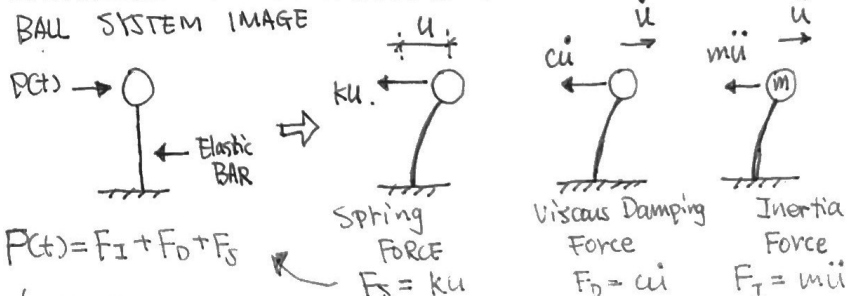


Motion Description.

$\ddot{u} \rightarrow u'' \rightarrow \frac{d}{dt} \left[\frac{du}{dt} \right] \rightarrow \text{Acceleration}$
 $\dot{u} \rightarrow u' \rightarrow \frac{du}{dt} \rightarrow \text{Velocity}$
 $u \rightarrow \text{Displacement}$

Unit
 mm/s^2 CAN feel it
 mm/s CAN see it
 mm

BALL SYSTEM IMAGE

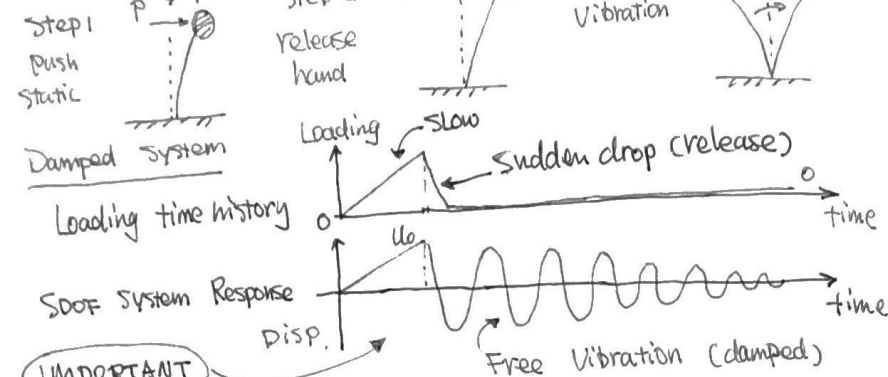


① $m\ddot{u} + ku = 0$	Free Vibration	Un-damped
② $m\ddot{u} + c\dot{u} + ku = 0$	Free Vibration	damped
③ $m\ddot{u} + ku = P(t)$	Forced Vibration	Un-damped
④ $m\ddot{u} + c\dot{u} + ku = P(t)$	Forced Vibration	damped.

FREE vibration

$$m\ddot{u} + k\dot{u} + cu = P(t) = 0 \quad [\text{damped}]$$

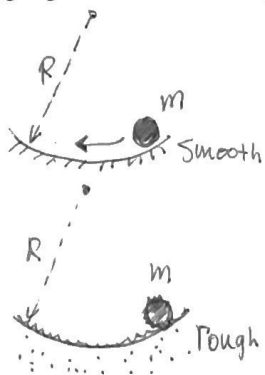
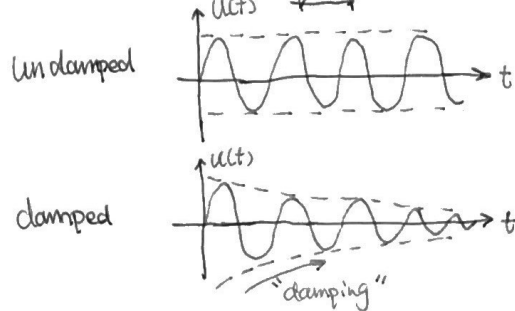
$$m\ddot{u} + k\dot{u} = P(t) = 0 \quad [\text{un-damped}]$$



IMPORTANT

Free Vibration	Show	System Properties: m, K, C
Push action	Show	System Properties: K
Push over	Show	System Properties: $K, F_y, \dots$

Free Vibration Type



Undamped system $\rightarrow T_1$ (Period)

damped system $\rightarrow T_1$ (Period) + ζ (damping Ratio).

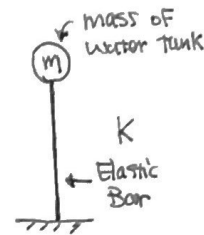
SDOF system

How to Simplify Structure to be SDOF system

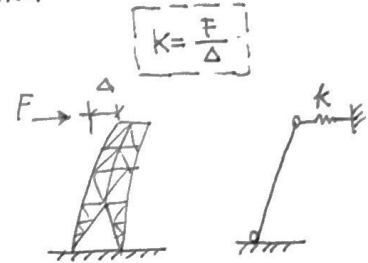
Example:



Water Tank



SDOF model



SDOF Spring Stiffness

[K] Value of Cantilever Beam

$$\Delta = \frac{P \cdot L^3}{3EI} \Rightarrow K = \frac{P}{\Delta} = \frac{3EI}{L^3}$$

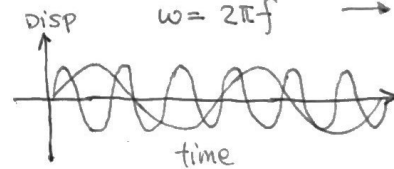
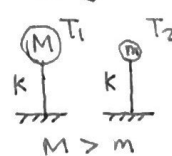
[K] Value of Simply Support Beam (SDOF) System

$$\Delta = \frac{PL^3}{48EI} \Rightarrow K = \frac{P}{\Delta} = \frac{48EI}{L^3}$$

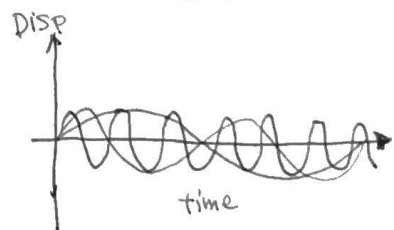
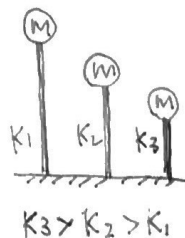
SDOF System Vertical Vibration

Fundamental Period

$$T_1 = 2\pi \sqrt{\frac{M}{K}}$$



$$T_1 > T_2$$



For Example

$$T_1 = 4.0 \text{ (Sec)}$$

$$T_2 = 2.0 \text{ (Sec)}$$

$$T_3 = 1.0 \text{ (Sec)}$$

$$\text{Mass} + \text{Stiffness} \rightarrow \text{Period.}$$

S DOF - FREE - VIBRATION

Basic Equation. $m\ddot{u} + ku = 0$

Assume. $u = \sin(\omega t)$

$$\dot{u} = \cos(\omega t) \cdot \omega$$

$$\ddot{u} = -\sin(\omega t) \cdot \omega^2$$

$$m \cdot \ddot{u} + ku = 0$$

$$\rightarrow m \cdot (-\omega^2) \cdot \sin \omega t + k \cdot \sin \omega t = 0$$

$$\rightarrow -m\omega^2 + k = 0$$

$$\rightarrow \omega = \sqrt{\frac{k}{m}}$$

$$\rightarrow T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}$$

Response of SDOF system Free-Vibration [Undamped]

Basic Equation $m\ddot{u} + ku = 0$

Assume $u = A_1 \cos \omega_n t + A_2 \sin \omega_n t$

General Solution

$$\dot{u}(t) = -A_1 \omega_n \sin \omega_n t + A_2 \omega_n \cos \omega_n t$$

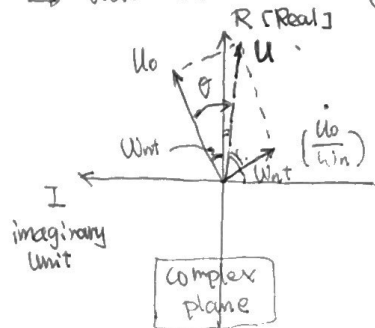
$$\ddot{u}(t) = -A_1 \omega_n^2 \cos \omega_n t - A_2 \omega_n^2 \sin \omega_n t$$

initial Condition $u_0 = u(0) \rightarrow$ initial Displacement
 $\dot{u}_0 = \dot{u}(0) \rightarrow$ initial Velocity

$$\dot{u}(0) = A_2 \omega_n \cdot (1) = A_2 \omega_n$$

$$u(0) = A_1 + A_2 \cdot (0) = A_1$$

$$\rightarrow u(t) = u_0 \cos \omega_n t + \left(\frac{\dot{u}_0}{\omega_n}\right) \sin \omega_n t \rightarrow \text{Final Solution}$$



$$u(t) = u_0 \cos \omega_n t + \left(\frac{\dot{u}_0}{\omega_n}\right) \sin \omega_n t$$

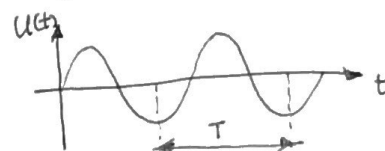
$$u(t) = U \cos(\omega_n t - \theta)$$

$$\tan \theta = \frac{\dot{u}_0 / \omega_n}{u_0}$$

$$U^2 = u_0^2 + \left(\frac{\dot{u}_0}{\omega_n}\right)^2$$

Derivative.

$$u = \sin \omega t$$



$$T = \frac{2\pi}{\omega}$$

Math Tips:

$$(\sin t)' = \cos t$$

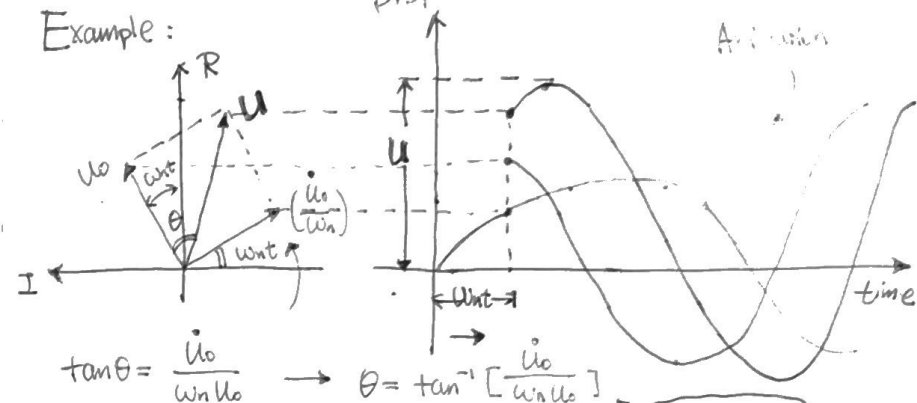
$$(\cos t)' = -\sin t$$

$$(uv)' = u \cdot v' + u' \cdot v$$

$$(au)' = a \cdot u' \quad a = \text{const.}$$

Undamped system

Example:



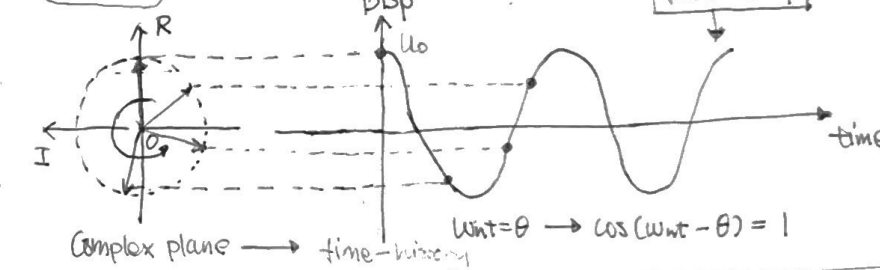
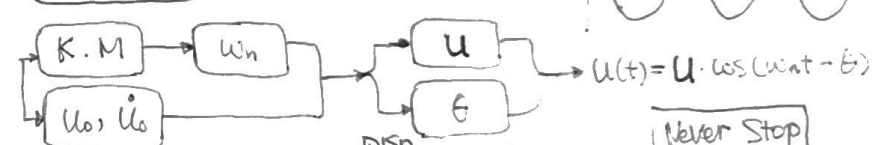
$$\tan \theta = \frac{\dot{u}_0}{\omega_n u_0} \rightarrow \theta = \tan^{-1} \left[\frac{\dot{u}_0}{\omega_n u_0} \right]$$

$$U = \sqrt{u_0^2 + \left(\frac{\dot{u}_0}{\omega_n}\right)^2}$$

$$u(t) = U \cos(\omega_n t - \theta) \quad \omega_n = \sqrt{\frac{k}{m}}$$

REMEMBER CHART

PROGRAM



S DOF - SYSTEM - FREE - VIBRATION - Damped - system.

Basic Equation. $m\ddot{u} + ku + c\dot{u} = 0$

$$\text{GIVEN } \omega_d = \omega_n \sqrt{1 - \zeta^2} \rightarrow \zeta : \text{damping Ratio}$$

General Solution

Assume: $u(t) = \exp(-\zeta \omega_n t) \cdot [A_1 \cos \omega_d t + A_2 \sin \omega_d t]$

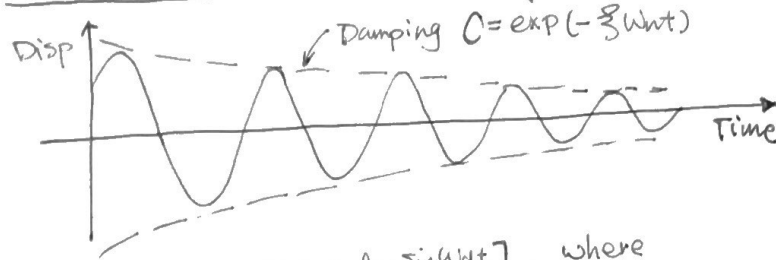
$$\dot{u}(0) = \dot{u}_0 \text{ initial Velocity}$$

$$u(0) = u_0 \text{ initial displacement}$$

Calculation

Math $\rightarrow$ Differential $\rightarrow$ Solution

Damped System. FREE-VIBRATION



The exponent applied to the base e

$$u(t) = C \cdot [A_1 \cos \omega_d t + A_2 \sin \omega_d t] \quad \text{where} \quad C = \exp(-\zeta \omega_n t)$$

damping Factor Free Vibration of undamped System.

Differential

$$u_0(0) = u_0 = \exp(-\zeta \omega_n \cdot 0) \cdot [A_1 \cdot 1 + A_2 \cdot 0] = 1 \times A_1 = A_1$$

$$\dot{u}_0(0) = \dot{u}_0 = [\dot{u}(t)]$$

$$\begin{aligned} &= [\exp(-\zeta \omega_n t) \cdot A_1 \cos \omega_d t]' \\ &+ [\exp(-\zeta \omega_n t) \cdot A_2 \sin \omega_d t]' \\ &= [\exp(-\zeta \omega_n t)]' [A_1 \cos \omega_d t] + [\exp(-\zeta \omega_n t)] [A_1 \cos \omega_d t]' \\ &+ [\exp(-\zeta \omega_n t)]' [A_2 \sin \omega_d t] + [\exp(-\zeta \omega_n t)] [A_2 \sin \omega_d t]' \\ &= (-\zeta \omega_n) \cdot \exp(-\zeta \omega_n t) \cdot A_1 \cos \omega_d t + \exp(-\zeta \omega_n t) \cdot A_2 \cdot \omega_d \cdot \cos \omega_d t \\ &= (-\zeta \omega_n) \cdot A_1 + \omega_d \cdot A_2 = \dot{u}_0 \quad \leftarrow \text{initial Velocity} \end{aligned}$$

$$\rightarrow A_2 = \frac{\dot{u}_0 + \zeta \omega_n \cdot u_0}{\omega_d}, \quad A_1 = u_0 \quad \leftarrow \text{initial Displacement.}$$

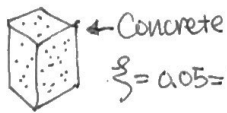
DAMPING RATIO ζ

$$C_{cr} = 2\sqrt{km} \quad \leftarrow \text{Critical Damping Constant}$$

$$C = \zeta \cdot C_{cr}$$

$$\rightarrow \zeta = C / C_{cr} \quad \text{damping Ratio}$$

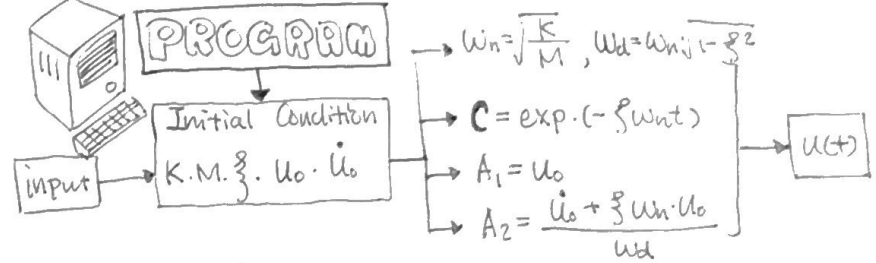
TYPICAL value:



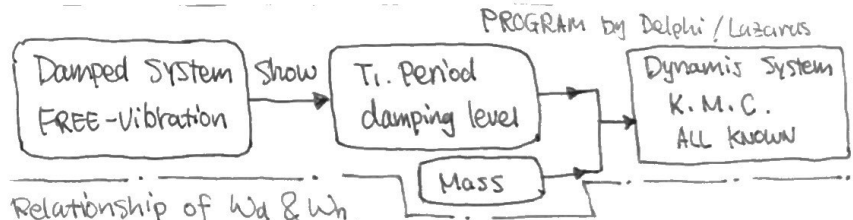
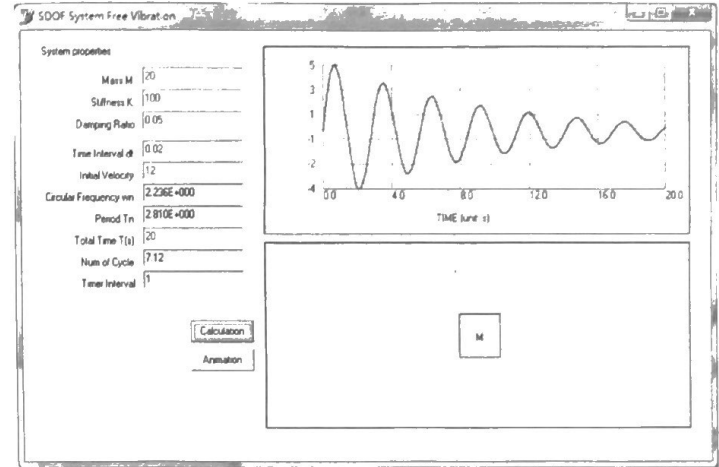
Concrete
 $\zeta = 0.05 = 5\%$



Steel
 $\zeta = 0.02 = 2\%$



$$\rightarrow u(t) = C \cdot [A_1 \cos \omega_d t + A_2 \sin \omega_d t]$$

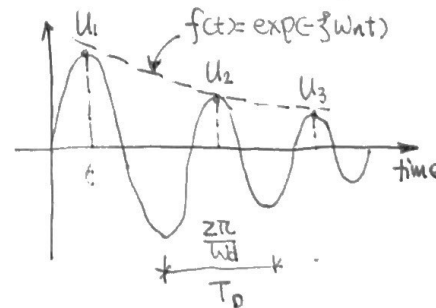


Relationship of ω_d & ω_n .

$$\text{Concrete } \zeta = 0.05 \quad \omega_d = \sqrt{1 - 0.05^2} \quad \omega_n = 0.974 \omega_n \rightarrow \omega_d \approx \omega_n$$

$$\text{Steel } \zeta = 0.02 \quad \omega_d = \sqrt{1 - 0.02^2} \quad \omega_n = 0.989 \omega_n$$

How to find out Damping Ratio



$$\frac{u_3}{u_1} = \frac{\exp[-\zeta \omega_n (t + 2T_0)]}{\exp[-\zeta \omega_n t]}$$

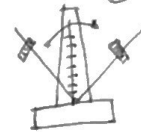
$$= \exp[-\zeta \omega_n \cdot 2 \cdot T_0]$$

$$\begin{aligned} \ln\left(\frac{u_3}{u_1}\right) &= -\zeta \omega_n \cdot 2 \cdot T_0 = -\zeta \cdot \omega_n \cdot 2 \cdot \frac{2\pi}{\omega_d} \\ &= -2\pi(2) \cdot \zeta \end{aligned}$$

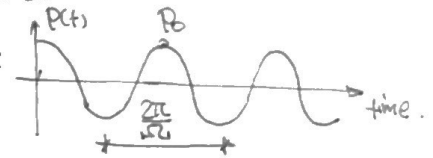
$$\text{Damping Ratio} \Rightarrow \zeta = \frac{1}{2\pi(2)} \ln\left(\frac{u_3}{u_1}\right)$$

HARMONIC VIBRATION

SDOF system
[undamped]



$$P(t) = P_0 \cdot \cos \Omega t$$



Undamped System → Basic Equation: $m\ddot{u} + ku = P(t) = P_0 \cdot \cos \Omega t$

Particular Solution → $u_p(t) = U \cdot \cos \Omega t$

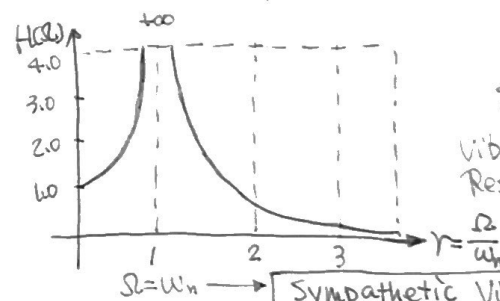
$$\begin{aligned} \dot{u}_p(t) &= -U \cdot \Omega \cdot \sin \Omega t \\ \ddot{u}_p(t) &= -U \cdot \Omega^2 \cdot \cos \Omega t \end{aligned} \rightarrow m \cdot (-U \cdot \Omega^2) + k \cdot U = P_0$$

$$\rightarrow U = \frac{P_0}{k - m\Omega^2} = \frac{P_0/k}{k/k - \frac{m}{k} \cdot \Omega^2} = \frac{U_0}{1 - \frac{m}{k} \Omega^2} = \frac{U_0}{1 - \frac{\Omega^2}{\omega^2}} = \frac{U_0}{1 - r^2}$$

$r = \Omega/\omega \rightarrow$ Circular Frequency Ratio $\rightarrow \frac{\text{Loading frequency}}{\text{Stru. frequency}}$

$U_0 = \frac{P_0}{k} \rightarrow$ Static deformation

$$H(\Omega) = \frac{U}{U_0} = \frac{U_0}{1 - r^2} \cdot \frac{1}{U_0} = \frac{1}{1 - r^2} \rightarrow \text{Frequency - Response Function.}$$



Solution

Solution = Particular + General
Vibration Response = steady Response + Free Vibration Response

$\Omega = \omega_n \rightarrow$ Sympathetic Vibration

$$u(t) = \left[\frac{U_0}{1 - r^2} \right] \cdot \cos \Omega t + A_1 \cdot \cos \omega_n t + A_2 \cdot \sin \omega_n t$$

Initial condition $\rightarrow$ $u(0) = 0 \rightarrow \left[\frac{P_0/k}{1 - r^2} \right] + A_1 = 0 \rightarrow A_1 = -\left[\frac{P_0/k}{1 - r^2} \right]$
 $\dot{u}(0) = 0 \rightarrow 0 + A_2 \cdot \omega_n = 0 \rightarrow A_2 = 0$

Damped system [SDOF] Harmonic Vibration

Basic Equation: $m\ddot{u} + c\dot{u} + ku = \boxed{P_0 \cdot \sin \Omega t}$ SIN

$$\ddot{u} + \frac{c}{m} \cdot \dot{u} + \frac{k}{m} u = \frac{P_0}{m} \cdot \sin \Omega t$$

$$\ddot{u} + 2\zeta \omega \cdot \dot{u} + \omega^2 u = \frac{P_0}{m} \cdot \sin \Omega t$$

Particular Solution $u(t) = G_1 \cdot \sin \Omega t + G_2 \cdot \cos \Omega t$ [Steady]

Velocity: $\dot{u}_p(t) = G_1 \cdot \Omega \cdot \cos \Omega t + (-G_2 \cdot \Omega) \cdot \sin \Omega t$

Acceleration: $\ddot{u}_p(t) = -G_1 \cdot \Omega^2 \cdot \sin \Omega t - G_2 \cdot \Omega^2 \cdot \cos \Omega t$

$$\rightarrow [-G_1 \cdot \Omega^2 - 2G_2 \cdot \zeta \omega \Omega + G_1 \cdot \omega^2 - \frac{P_0}{m}] \cdot \sin \Omega t$$

$$= [G_2 \omega^2 - 2G_1 \zeta \omega \Omega - G_2 \cdot \omega^2] \cdot \cos \Omega t$$

$$\rightarrow \begin{cases} -G_1 \Omega^2 - 2G_2 \zeta \omega \Omega + G_1 \omega^2 - \frac{P_0}{m} = 0 \\ G_2 \omega^2 - 2G_1 \zeta \omega \Omega - G_2 \omega^2 = 0 \end{cases}$$

$$\rightarrow \begin{cases} G_1 = \frac{P_0}{K} \cdot \frac{1-r^2}{(1-r^2)^2 + (2\zeta r)^2} \\ G_2 = \frac{P_0}{K} \cdot \frac{-2\zeta r}{(1-r^2)^2 + (2\zeta r)^2} \end{cases}$$

Particular Solution

$$u_p(t) = \frac{P_0}{K} \cdot \frac{1}{(1-r^2)^2 + (2\zeta r)^2} \cdot [(1-r^2) \cdot \sin \Omega t - 2\zeta r \cdot \cos \Omega t]$$

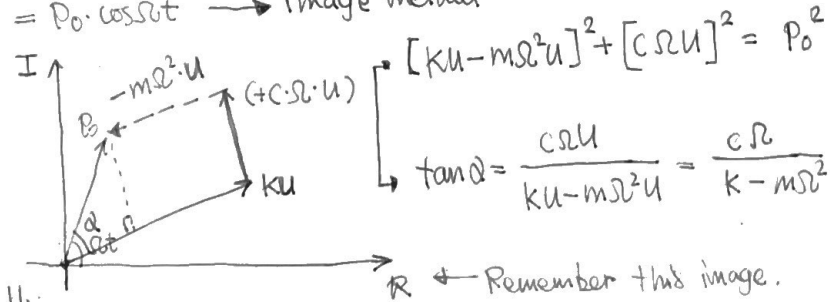
Easy Method Particular Solution $u_p(t) = U \cdot \cos(\Omega t - \alpha)$

$$\dot{u}_p(t) = -U \cdot \Omega \sin(\Omega t - \alpha)$$

$$\ddot{u}_p(t) = -U \cdot \Omega^2 \cos(\Omega t - \alpha)$$

$$m \cdot (-U \cdot \Omega^2) \cos(\Omega t - \alpha) + (-U \cdot \Omega) \cdot \sin(\Omega t - \alpha) \cdot c + K \cdot U \cdot \cos(\Omega t - \alpha) = P_0 \cdot \cos \Omega t$$

→ image method

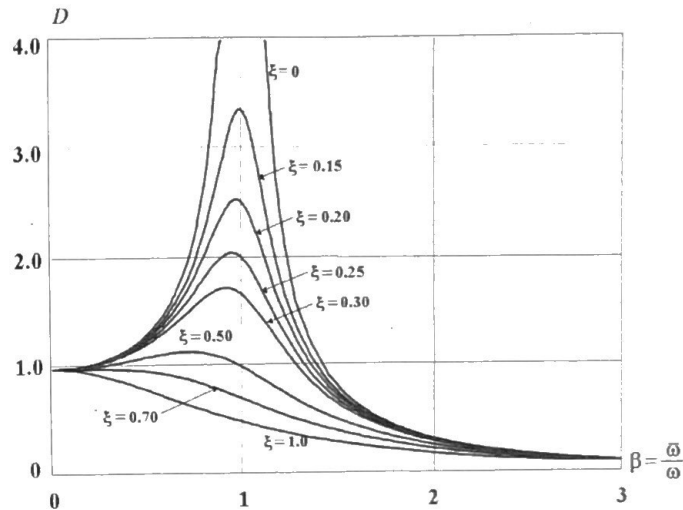


$$U^2 \cdot [k - m\Omega^2]^2 + [c\Omega]^2 \cdot U^2 = P_0^2$$

$$U = P_0 \cdot \frac{1}{\sqrt{(k - m\Omega^2)^2 + (c\Omega)^2}} \quad U_0 = P_0/k$$

$$\frac{U}{U_0} = \frac{P_0}{\sqrt{(k - m\Omega^2)^2 + (c\Omega)^2}} \cdot \frac{K}{P_0} = \frac{K}{\sqrt{(k - m\Omega^2)^2 + (c\Omega)^2}} \quad \text{Steady Result.}$$

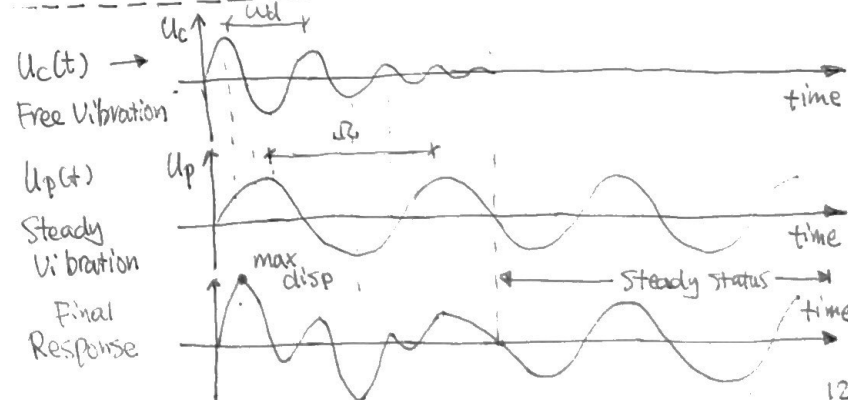
$$= \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = H(\Omega) \rightarrow \text{Frequency-Response Function.}$$



$D = H(\Omega) \quad \beta = r = \frac{\bar{\omega}}{\omega} = \frac{\Omega}{\omega}$. Damping Ratio help to Reduce Sympathetic Vibration

Response Displacement: ★

$$u(t) = u_c(t) + u_p(t) = \exp(-\zeta \omega t) \cdot (A_1 \sin \omega_d t + A_2 \cos \omega_d t) + \frac{P_0}{K} \cdot \frac{1}{(1-r^2)^2 + (2\zeta r)^2} \cdot [(1-r^2) \cdot \sin \Omega t - 2\zeta r \cdot \cos \Omega t]$$





PROGRAM Lazarus or Delphi.

SDOF System. Harmonic Vibration.

$$m\ddot{u} + c\dot{u} + ku = P_0 \sin \omega t$$

Response Equation

$$u(t) = u_c(t) = \exp(-\xi \omega_n t) \cdot [A_1 \sin \omega_n t + A_2 \cos \omega_n t] + u_p(t) = \frac{P_0}{K} \cdot \frac{1}{(1-r^2)^2 + (2\xi r)^2} \cdot [(1-r^2) \sin \omega t - 2\xi r \cos \omega t]$$

$$\dot{u}(0) = [\exp(-\xi \omega_n t) \cdot A_1 \sin \omega_n t + [\exp(-\xi \omega_n t) \cdot A_2 \cos \omega_n t]]'$$

$$= \exp(-\xi \omega_n t) \cdot A_1 \cdot \omega_n \cos \omega_n t + \exp(-\xi \omega_n t) \cdot (-\xi \omega_n) \cdot A_2 \cdot \omega_n \sin \omega_n t + 0$$

$$u_p(\omega) = (1-r^2) \cdot \Omega \cdot [D] \quad \text{where } D = \frac{P_0}{K} \cdot \frac{1}{(1-r^2)^2 + (2\xi r)^2}$$

$$u_0(\omega) = [A_1 \cdot \omega_n - \xi \omega_n \cdot A_2] + (1-r^2) \cdot \Omega \cdot D = 0$$

$$u(\omega) = (1-r^2) \cdot [A_2 \cdot 1] + [D] \cdot [-2\xi r \cdot 1] = 0$$

$$u(\omega) = A_2 + D \cdot (-2\xi r) = 0 \Rightarrow A_2 = (2\xi r) \cdot D$$

$$\dot{u}(\omega) = A_1 \cdot \omega_n - \xi \omega_n \cdot (2\xi r) \cdot D + (1-r^2) \cdot \Omega \cdot D = 0$$

$$\Rightarrow A_1 = \frac{\xi \omega_n \cdot (2\xi r) \cdot D - (1-r^2) \cdot \Omega \cdot D}{\omega_n}$$

ZERO START!

PROGRAM

Initial Condition

$$K, M, \xi, P_0, \Omega, u_0=0, \dot{u}_0=0$$

$$\omega_n = \sqrt{K/M}$$

$$\omega_d = \omega_n \cdot \sqrt{1-\xi^2}$$

$$r = \Omega / \omega_n$$

$$D = \frac{P_0}{K} \cdot \frac{1}{(1-r^2)^2 + (2\xi r)^2}$$

$$A_2 = (2\xi r) \cdot D$$

$$A_1 = \frac{\xi \omega_n \cdot (2\xi r) \cdot D - (1-r^2) \cdot \Omega \cdot D}{\omega_d}$$

$$u(t) = \exp(-\xi \omega_n t) \cdot [A_1 \sin \omega_d t + A_2 \cos \omega_d t] + D \cdot [(1-r^2) \sin \Omega t - 2\xi r \cos \Omega t]$$

$$\text{Basic Equation: } m\ddot{u} + c\dot{u} + ku = P_0 \cos \omega t$$

$$u(t) = u_c(t) + u_p(t) = \exp(-\xi \omega_n t) \cdot [A_1 \sin \omega_d t + A_2 \cos \omega_d t] + U \cdot \cos(\Omega t - \alpha)$$

$$u(0) = u_0 = 0 = (1) \cdot [0 + A_2] + U \cdot \cos(-\alpha) = 0$$

$$\Rightarrow A_2 = -U \cos(-\alpha) = -U \cdot \cos(\alpha)$$

$$\dot{u}(0) = \exp(-\xi \omega_n t) \cdot [A_1 \cdot \omega_d \cdot (1)] + (-\xi \omega_n) \cdot (1) \cdot [A_2 \cdot (1)] + [U \cos(\Omega t - \alpha)]'$$

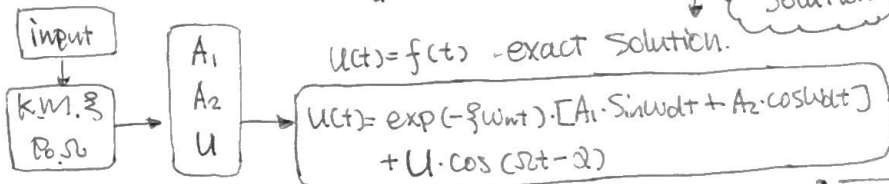
$$= A_1 \cdot \omega_d - A_2 \cdot (\xi \omega_n) - U \cdot \Omega \cdot \sin(\Omega t - \alpha)$$

$$= A_1 \cdot \omega_d - A_2 \cdot \xi \omega_n + U \cdot \Omega \cdot \sin(\alpha) = 0$$

$$= A_1 \cdot \omega_d + U \cdot \cos \alpha \cdot \xi \omega_n + U \cdot \Omega \cdot \sin \alpha = 0$$

$$\Rightarrow A_1 = \frac{-(U \cdot \Omega \cdot \sin \alpha + U \cdot \xi \cdot \omega_n \cdot \cos \alpha)}{\omega_d}$$

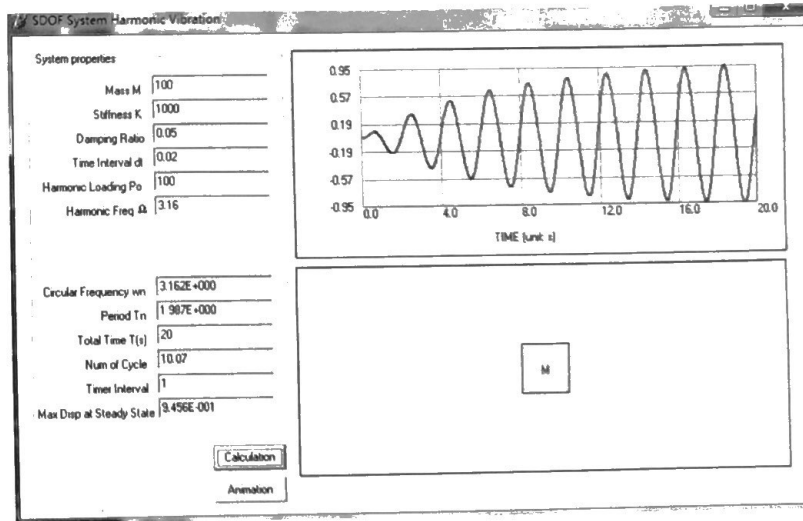
Elegant Solution



$u(t) = f(t)$ - exact Solution.

Program: SDOF System. Harmonic Vibration Calculation.

$$\text{Basic Equation} \rightarrow m\ddot{u} + c\dot{u} + ku = P_0 \cos \omega t$$



SDOF

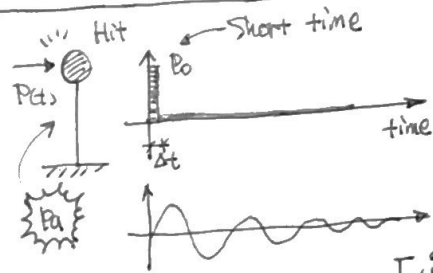
SYSTEM VIBRATION
NUMERICAL SOLUTION

Duhamel Integration

Arbitrary loading $P(t)$

→ Impulse loading → Free Vibration
→ Integration

BIG
Table



$I = P_0 \Delta t$ (Unit Impulse)

Free Vibration

$$m\ddot{u} + c\dot{u} + ku = 0$$

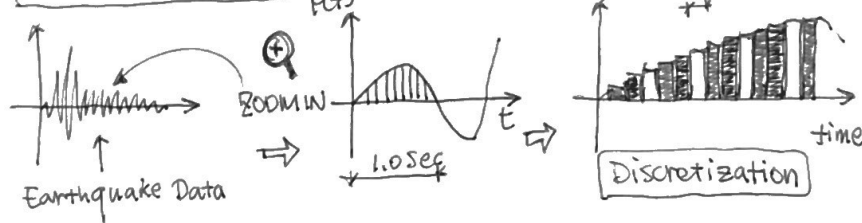
$$u_0 = 0 \quad \dot{u}_0 = \frac{P_0 \Delta t}{m}$$

Solution $u(t) = \exp(-\xi \omega_n t) \cdot \left[\frac{\dot{u}_0 + \dot{u}_0 \xi \omega_n}{\omega_d} \sin \omega_d t + u_0 \cos \omega_d t \right]$

Free Vibration $= \exp(-\xi \omega_n t) \cdot \frac{P_0 \Delta t}{m \omega_d} \sin \omega_d t$

important $= \exp(-\xi \omega_n t) \cdot \frac{I}{m \omega_d} \sin \omega_d t$

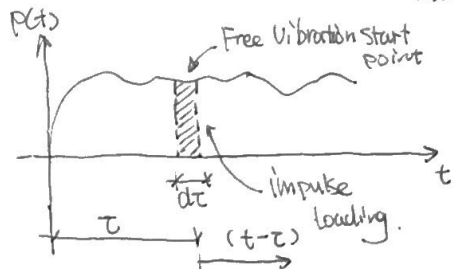
Time-history Data



Particular Solution

$U_p(t) = \int_0^t U(t) dt$ ← math tips: Convolution Integration method.

$$= \int_0^t \left[\exp(-\xi \omega_n(t-\tau)) \frac{P(\tau)}{m \omega_d} \sin \omega_d(t-\tau) \right] d\tau$$



Particular Solution

Oh! Math hand Calc

$$U_p(t) = \int_0^t \frac{P(\tau)}{m \omega_d} \cdot \exp[-\xi \omega_n(t-\tau)] \cdot \sin \omega_d(t-\tau) d\tau$$

$$= \frac{1}{m \omega_d} \int_0^t P(\tau) \cdot \frac{\exp(\xi \omega_n \tau)}{\exp(\xi \omega_n t)} \cdot [\cos \omega_d \tau \cdot \sin \omega_d t - \cos \omega_d t \cdot \sin \omega_d \tau] d\tau$$

$$= \frac{\sin \omega_d t}{m \omega_d} \int_0^t P(\tau) \cdot \frac{\exp(\xi \omega_n \tau)}{\exp(\xi \omega_n t)} [\cos \omega_d \tau] d\tau$$

$$- \frac{\cos \omega_d t}{m \omega_d} \int_0^t P(\tau) \cdot \frac{\exp(\xi \omega_n \tau)}{\exp(\xi \omega_n t)} [\sin \omega_d \tau] d\tau$$

$$= \sin \omega_d t \cdot \left[\frac{1}{m \omega_d} \int_0^t P(\tau) \cdot \frac{\exp(\xi \omega_n \tau)}{\exp(\xi \omega_n t)} \cdot \cos \omega_d \tau d\tau \right] \rightarrow A$$

$$- \cos \omega_d t \cdot \left[\frac{1}{m \omega_d} \int_0^t P(\tau) \cdot \frac{\exp(\xi \omega_n \tau)}{\exp(\xi \omega_n t)} \cdot \sin \omega_d \tau d\tau \right] \rightarrow B$$

$$= A \cdot \sin \omega_d t + B \cdot \cos \omega_d t$$

$$A = \frac{1}{m \omega_d} \int_0^t P(\tau) \cdot \frac{\exp(\xi \omega_n \tau)}{\exp(\xi \omega_n t)} \cdot \cos \omega_d \tau d\tau$$

$$B = \frac{1}{m \omega_d} \int_0^t P(\tau) \cdot \frac{\exp(\xi \omega_n \tau)}{\exp(\xi \omega_n t)} \cdot \sin \omega_d \tau d\tau$$

$$y = P(\tau) \cdot \cos \omega_d \tau \quad y_N = P(\tau_N) \cdot \cos \omega_d \tau_N$$

Fast Integration.

$$A = \frac{1}{m \omega_d} \int_0^t \frac{\exp(\xi \omega_n \tau)}{\exp(\xi \omega_n t)} \cdot y(\tau) d\tau$$

**SIMPSON'S
RULE**

$$A_N = A_{N-2} \cdot \exp(-2\xi \omega_n \Delta t)$$

$$+ \frac{\Delta \tau}{3 m \omega_d} \cdot [y_{N-2} \cdot \exp(-2\xi \omega_n \Delta t) + 4y_{N-1} \cdot \exp(-\xi \omega_n \Delta t) + y_N]$$

$$N = 2, 4, 6, \dots$$

$$Z = P(\tau) \cdot \sin \omega_d \tau \quad Z_N = P(\tau_N) \cdot \sin \omega_d \tau_N$$

$$B = \frac{1}{m \omega_d} \int_0^t \frac{\exp(\xi \omega_n \tau)}{\exp(\xi \omega_n t)} \cdot Z(\tau) d\tau$$

$$B_N = B_{N-2} \cdot \exp(-2\xi \omega_n \Delta t)$$

$$+ \frac{\Delta \tau}{3 m \omega_d} \cdot [Z_{N-2} \cdot \exp(-2\xi \omega_n \Delta t) + 4Z_{N-1} \cdot \exp(-\xi \omega_n \Delta t) + Z_N]$$

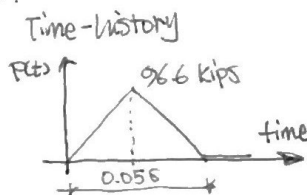
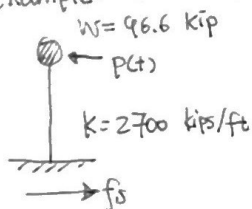
$$N = 2, 4, 6, \dots$$

Duhamel

Integration

CAN USE EXCEL TABLE
Simple
EXCEL → PROGRAM.

Example: SDOF SYSTEM.



damping Ratio
 $\xi = 0.05$

$$m = \frac{W}{g} = 30 \text{ rad/s} \quad \Delta t = 0.005 \text{ sec} \quad F = \frac{\Delta t}{3mW} = 1.852 \times 10^{-5}$$

$$M_1 = 4 \exp(-\xi \omega \Delta t) = 3.97 \quad M_2 = \exp(-2\xi \omega \Delta t) = 0.985$$

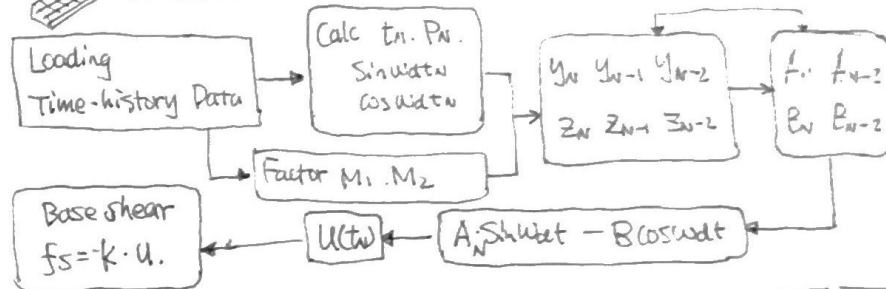
TABLE E6-2
Numerical Duhamel integral analysis including damping

N	t_n sec	P_n kips (1)	$\sin 30 t_n$ (2)	$\cos 30 t_n$ (3)	y_n (1) × (3) kips (4)	y_{n-1} (5)	y_{n-2} (6)	$M_1 \times$ (5) (7)	$M_2 \times$ (6) × (9) (8)	$\frac{A_{n-2}}{F}$ (9)	$\frac{A_n}{F}$ (4) × (7) + (8) (10)	y_n (1) × (2) kips (11)	y_{n-1} (12)	y_{n-2} (13)
0	0.000	0	0	1.000	0	—	—	—	—	—	0	0	—	—
1	0.005	19.32	0.149	0.989	19.1	0	—	—	—	—	2.88	0	—	—
2	0.010	38.64	0.296	0.955	36.9	19.1	0	75.8	0	0	112.7	11.4	2.88	0
3	0.015	57.96	0.435	0.900	52.2	36.9	19.1	—	—	—	—	25.2	11.4	2.88
4	0.020	77.28	0.565	0.825	63.8	52.2	36.9	207.2	147.4	112.7	418.4	43.7	25.2	11.4
5	0.025	96.60	0.682	0.732	70.7	63.8	52.2	—	—	—	—	65.9	43.7	25.2
6	0.030	77.28	0.783	0.622	48.1	70.7	63.8	280.7	475.0	418.4	803.8	60.5	65.9	43.7
7	0.035	57.96	0.867	0.498	28.9	48.1	70.7	—	—	—	—	50.3	60.5	43.7
8	0.040	38.64	0.932	0.362	14.0	28.9	48.1	114.7	839.1	803.8	967.8	36.0	50.3	43.7
9	0.045	19.32	0.976	0.219	4.23	14.0	28.9	—	—	—	—	18.9	36.0	50.3
10	0.050	0	0.997	0.0707	0	4.23	14.0	16.8	967.1	967.8	983.9	0	18.9	36.0
11	0.055	0	0.997	-0.079	0	0	4.23	—	—	—	—	0	0	18.9
12	0.060	0	0.974	-0.227	0	0	0	0	969.1	983.9	969.1	0	0	0
13	0.065	0	0.929	-0.370	0	0	0	—	—	—	—	0	0	0
14	0.070	0	0.863	-0.505	0	0	0	0	954.6	969.1	0	0	0	0
15	0.075	0	0.778	-0.628	0	0	0	—	—	—	—	0	0	0
16	0.080	0	0.675	-0.737	0	0	0	0	940.3	954.6	940.3	0	0	0
17	0.085	0	0.558	-0.830	0	0	0	—	—	—	—	0	0	0
18	0.090	0	0.427	-0.904	0	0	0	0	926.2	940.3	926.2	0	0	0



PROGRAM
PROCEDURE

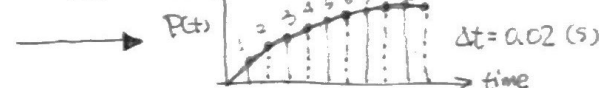
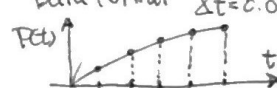
Delphi
Excel.



Math [Z] Tips: Linear - Interpolation.

Data Format $\Delta t = 0.045$

$\Delta t = 0.025$



$M_1 \times$ (12) (14)	$M_2 \times$ (13) × (16) (15)	$\frac{B_{n-2}}{F}$ (16)	$\frac{B_n}{F}$ (17)	$(11) \times (14) \times (15)$ (18)	$(10) \times (2)$ (19)	$(17) \times (3)$ (20)	$(18) \times (19)$ (21)	v_n $F \times (20)$ ft (21)	f_{s_n} $k \times (21)$ kips (22)
—	—	—	0	—	0	0	0	0	0
—	—	—	—	—	—	—	—	—	—
11.4	0	0	22.8	33.3	21.8	11.5	0.0002	0.58	—
—	—	—	—	—	—	—	—	—	—
100.0	33.7	22.8	177.4	236	146	90	0.0017	4.50	—
—	—	—	—	—	—	—	—	—	—
261.6	217.8	177.4	539.9	629	336	293	0.0054	14.65	—
—	—	—	—	—	—	—	—	—	—
199.7	591.4	539.9	827.1	902	299	603	0.0112	30.2	—
—	—	—	—	—	—	—	—	—	—
75.0	850.1	827.1	925.1	981	65.4	915	0.0169	45.8	—
—	—	—	—	—	—	—	—	—	—
0	911.2	925.1	911.2	900	-206	1106	0.0205	55.4	—
—	—	—	—	—	—	—	—	—	—
0	897.5	911.2	897.5	824	-453	1277	0.0236	63.9	—
—	—	—	—	—	—	—	—	—	—
0	884.0	897.5	884.0	635	-651.5	1286	0.0238	64.3	—
—	—	—	—	—	—	—	—	—	—
0	870.7	884.0	870.7	395	-787	1182	0.0219	59.1	—

Famous Ground Motion Acceleration Data. (Unit: gal)

<http://dino.chen.com/GMData/GM-ELCentrul.txt>

